The Geometric Invariants Ω^n of a Product of Groups

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Abstract

We compute the geometric invariants Ω^n of a product $G \times H$ of groups in terms of $\Omega^n(G)$ and $\Omega^n(H)$. This gives a sufficient condition in terms of $\Omega^n(G)$ and $\Omega^n(H)$ for a normal subgroup of $G \times H$ with abelian quotient to be of type F_n. We give an example involving the direct product of the Baumslag-Solitar group $BS_{1,2}$ with itself.

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1 Introduction

Let G and H be groups. We compute the geometric invariants $\Omega^n(G \times H)$ in terms of $\Omega^n(G)$ and $\Omega^n(H)$. The invariants Ω^n were defined in [6] and are analogs of the Bieri-Neumann-Strebel-Renz invariants Σ^n defined in [3] for $n = 1$ and in [4] for $n \geq 2$. We recall these definitions here for $n = 1$; the full definitions are given in § 2.1 and § 2.2.

Let G be a finitely generated group with generating set $\mathcal{X}$. The set $\text{Hom}(G, \mathbb{R})$ of homomorphisms from G to the additive group of reals is a real vector space with dimension equal to the $\mathbb{Z}$-rank of the abelianization of G, so $\text{Hom}(G, \mathbb{R}) \cong \mathbb{R}^m$ for some m. A *geodesic ray* in $\mathbb{R}^m$ is a continuous function $\gamma : [0, \infty) \to \mathbb{R}^m$ such that for all $a, b \in [0, \infty)$, $\|\gamma(a) - \gamma(b)\| = |a - b|$. Two geodesic rays γ and γ' are *equivalent* if there exists $k \geq 0$ such that for all $t \in [0, \infty)$, $\|\gamma(t) - \gamma'(t)\| = k$. Denote by $\partial_\infty \mathbb{R}^m$ the set of asymptotic equivalence classes of geodesic rays in $\mathbb{R}^m$. Let $e \in \partial_\infty \mathbb{R}^m$ and let γ be a geodesic ray defining e. For each $s > 0$, denote by $H_{\gamma,s}$ the half-space in $\mathbb{R}^m$ whose boundary is orthogonal to γ with $H_{\gamma,s} \cap \gamma([0, \infty)) = \gamma([s, \infty))$. Let X denote the Cayley graph of G with respect to $\mathcal{X}$. Since the $\mathbb{Z}$-rank of the abelianization of G is m, there is a homomorphism $\pi : G \to \mathbb{Z}^m$. Define $h : X \to \mathbb{R}^m$ by: $h(g) := \pi(g)$ for all vertices $g \in X$, and extend linearly on edges. For each $s \geq 0$, let $X_{\gamma,s} := h^{-1}(H_{\gamma,s})$. The direction $e \in \Sigma^1(G)$ if and only if for every $s \geq 0$, there exists $\lambda = \lambda(s) \geq 0$ such that any two points $u, v \in X_{\gamma,s}$ can be joined by a path in $X_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$.

In the compactified space $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$, the compactified half-spaces play the role of neighborhoods of the point $e \in \partial_\infty \mathbb{R}^m$, but this gives an unsatisfactory topology to $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$ (see Figure 1). From the point of view of topology, it is more natural to have a similar definition to $\Sigma^1(G)$ using "ordinary" neighborhoods of e. A basis for these neighborhoods consists of "truncated cones". For each $s \geq 0$ and each geodesic ray γ, define the *truncated cone* $C_{\gamma,s} := Cone_\theta(\gamma) \cap H_{\gamma,s}$ where $Cone_\theta(\gamma)$ is the closed cone of angle θ and vertex $\gamma(0)$, and $\theta := \arctan(\frac{1}{s})$ if $s > 0$ and $\theta := \frac{\pi}{2}$ if $s = 0$. Let $Y_{\gamma,s} := h^{-1}(C_{\gamma,s})$. We say that $e \in \Omega^1(G)$ if and only if there exists $s_0 \geq 0$ such that for each $s \geq s_0$, there exists $\lambda = \lambda(s) \geq 0$ such that any two points $u, v \in Y_{\gamma,s}$ can be joined by a path in $Y_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$.

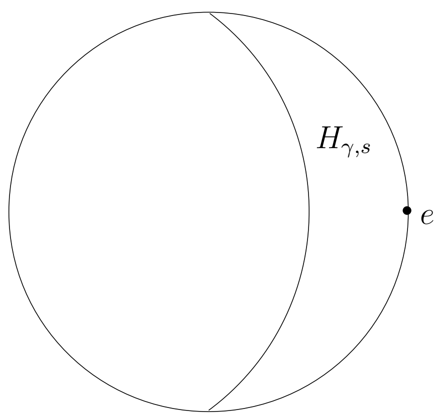

Figure 1: The compactified half-spaces in $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$ give these "crescent moon" neighborhoods of e.

Example: Let G be the Baumslag-Solitar group $BS_{1,2} \cong \langle x, s | s^2 = x^{-1}sx \rangle$. Figure 2 shows a portion of the Cayley graph X of G. Since $s = s^{-1}x^{-1}sx$, any homomorphism from G to $\mathbb{R}$ would have to take s to 0. Thus, the vector space $\text{Hom}(G, \mathbb{R}) \cong \mathbb{R}$ with basis $x \mapsto 1$. Let γ be a geodesic ray defining the direction ∞. For each $t \in \mathbb{Z}$, $X_{\gamma,t}$ is the portion of X lying above the level $x = t$ (see Figure 2). This is connected, so $\infty \in \Sigma^1(G)$ and $\infty \in \Omega^1(G)$. Let $-\gamma$ be a geodesic ray defining the direction $-\infty$. For each $t \in \mathbb{Z}$, $X_{-\gamma,t}$ is the portion of X lying below the level $x = t$ which is disconnected. Therefore, $\Sigma^1(BS_{1,2}) = \{\infty\}$ and $\Omega^1(BS_{1,2}) = \{\infty\}$.

We prove our main result, Theorem 3.8, in § 3.3. That theorem states that $\Omega^n(G \times H) = \Omega^n(G) * \Omega^n(H)$. Along the way to proving this we recall some ideas about spherical joins in § 3.1 and about contractions toward $e \in \partial_\infty \mathbb{R}^m$ in § 3.2.

It should be mentioned that it is an open problem to express $\Sigma^n(G \times H)$ in terms of the Σ-invariants of G and the Σ-invariants of H. Robert Bieri has conjectured the following (denote by $\Sigma^n_c(G)$ the complement of $\Sigma^n(G)$):

Conjecture 1.1 $\Sigma^n_c(G \times H) = \bigcup_{i=0}^{n} \Sigma^i_c(G) * \Sigma^{n-i}_c(H)$

A proof of one of the inclusions of Conjecture 1.1 is due to H. Meinert (unpublished).

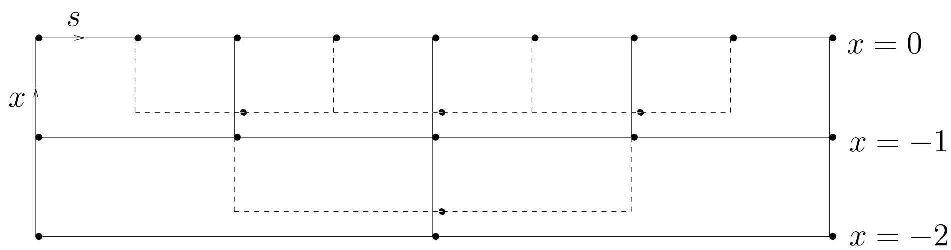

Figure 2: This is a piece of the Cayley graph of $BS_{1,2}$. The dotted lines indicate that the lines are behind the solid lines. The Cayley graph branches like a tree. For $a \in \{-2,-1,0\}$, the $x = a$ on the right indicates that, at that level, the exponent sum of the x's is a.

Theorem 1.2 $\Sigma_c^n(G \times H) \subseteq \bigcup_{i=0}^{n} \Sigma_c^i(G) * \Sigma_c^{n-i}(H)$

In § 4, we show that Theorem 3.8 is a consequence of Conjecture 1.1. We use Theorem 3.8 in § 5 to give a sufficient condition for a normal subgroup of $G \times H$ above the commutator subgroup to be of type F_n (a group G has *type* F_n if G has a $K(G,1)$ complex with a finite n-skeleton). We end with an example showing that a certain subgroup of $BS_{1,2} \times BS_{1,2}$ is finitely generated.

2 The Geometric Invariants Σ^n and Ω^n

Let n be a non-negative integer, and let G be a group of type F_n. In this section, we define two invariants of G:

1. the Bieri-Neumann-Strebel-Renz (or BNSR) invariants $\Sigma^n(G)$, and
2. the invariants $\Omega^n(G)$.

2.1 The BNSR invariants $\Sigma^n(G)$

For some $m \in \mathbb{Z}_{\geq 0}$, $\mathrm{Hom}(G,\mathbb{R}) \cong \mathbb{R}^m$. Let $e \in \partial_\infty \mathbb{R}^m$, and let γ be a geodesic ray defining e. Associated to γ is the function $\beta_\gamma : \mathbb{R}^m \to \mathbb{R}$ defined by $\beta_\gamma(a) := \langle u_e, a - \gamma(0)\rangle / \|u_e\|$ where $\langle \cdot,\cdot \rangle$ is the Euclidean inner product, $\|\cdot\|$ is the norm, and u_e is a vector (at 0) pointing toward e. For each $s \in \mathbb{R}$, let $H_{\gamma,s} := \beta_\gamma^{-1}([s,\infty))$. Each $H_{\gamma,s}$ is a closed half-space orthogonal to γ.

Since the $\mathbb{Z}$-rank of the abelianization of G is m, there exists an epimorphism $\pi : G \to \mathbb{Z}^m$, and thus there is a cocompact action $\rho : G \to \mathrm{Transl}(\mathbb{R}^m)$ of G on $\mathbb{R}^m$ by translations[1]. Pick an n-dimensional $(n-1)$-connected CW complex X on which G acts freely as a group of

[1]Denote by $\mathrm{Transl}(\mathbb{R}^m)$ the group of translations of $\mathbb{R}^m$

cell permuting homeomorphisms with $G\backslash X$ a finite complex. Choose a G-map $h : X \to \mathbb{R}^m$, and for each $s \in \mathbb{R}$, denote by $X_{\gamma,s}$ the largest subcomplex of X contained in $h^{-1}(H_{\gamma,s})$. The action ρ is *controlled* $(n-1)$*-connected* (or CC^{n-1}) *in the direction* e if for every $s \in \mathbb{R}$ and every $-1 \leq p \leq n-1$, there exists $\lambda = \lambda(s) \geq 0$ such that every map[2] $f : S^p \to X_{\gamma,s}$ can be extended to a map $\hat{f} : B^{p+1} \to X_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$. Define the *BNSR geometric invariants* of G to be $\Sigma^n(G) := \{e \in \partial_\infty \mathbb{R}^m | \rho \text{ is } CC^{n-1} \text{ in the direction } e\}$.

2.2 The invariants $\Omega^n(G)$

There are invariants analogous to $\Sigma^n(G)$ that replace half-spaces with "truncated cones". Let $e \in \partial_\infty \mathbb{R}^m$, and let γ be a geodesic ray defining e. For each $s \geq 0$, define the *truncated cone* $C_{\gamma,s} := Cone_\theta(\gamma) \cap H_{\gamma,s}$ where:

1. $\theta := \arctan(\frac{1}{s})$ if $s > 0$ and $\theta := \frac{\pi}{2}$ if $s = 0$, and
2. $Cone_\theta(\gamma)$ is the closed cone of angle θ and vertex $\gamma(0)$.

Choose X and h as before. Denote by $Y_{\gamma,s}$ the largest subcomplex of X contained in $h^{-1}(C_{\gamma,s})$. The action ρ is *bounded* $(n-1)$*-connected* (or BC^{n-1}) *in the direction* e if there exists $s_0 \geq 0$ such that for every $s \geq s_0$ and each $-1 \leq p \leq n-1$, there exists $\lambda = \lambda(s) \geq 0$ such that every map $f : S^p \to Y_{\gamma,s}$ can be extended to a map $\hat{f} : B^{p+1} \to Y_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$. Define $\Omega^n(G) := \{e \in \partial_\infty \mathbb{R}^m | \rho \text{ is } BC^{n-1} \text{ in the direction } e\}$.

The following theorem relates the invariants $\Sigma^n(G)$ and $\Omega^n(G)$.

Theorem 2.1 *[6, Theorem 3.1] Let $e \in \partial_\infty \mathbb{R}^m$. Then $e \in \Omega^n(G)$ if and only if $e' \in \Sigma^n(G)$ for every e' in an open $\frac{\pi}{2}$-neighborhood of e.*

Given $\Sigma^n(G)$, we can completely determine $\Omega^n(G)$: for each $e \in \partial_\infty \mathbb{R}^m$, $e \in \Omega^n(G)$ if and only if the open $\frac{\pi}{2}$-neighborhood of e is in $\Sigma^n(G)$. However, it is not the case that $\Omega^n(G)$ completely determines $\Sigma^n(G)$. For each $e \in \partial_\infty \mathbb{R}^m$, if there exists $e' \in \Omega^n(G)$ in the open $\frac{\pi}{2}$-neighborhood of e, then $e \in \Sigma^n(G)$, but if $\Omega^n(G)$ and the open $\pi/2$-neighborhood of e are disjoint, this does not imply that $e \in \Sigma^n_c(G)$. Examples of such groups are given in [6, § 1.3].

3 The Main Result

In preparation for our main result, Theorem 3.8, we need to recall two ideas: (1) the spherical join, $\mathbb{S}^k * \mathbb{S}^m$, of two spheres, and (2) contractions toward $e \in \partial_\infty \mathbb{R}^m$.

[2]Throughout this paper, by a map we will mean a continuous function.

3.1 Spherical joins

In this section, we review the spherical join of two spheres and some related ideas as discussed in [5, pp. 63-64]. The *spherical join* of the spheres $\mathbb{S}^k$ (with metric d_k) and $\mathbb{S}^m$ (with metric d_m) is $\mathbb{S}^k * \mathbb{S}^m := ([0, \frac{\pi}{2}] \times \mathbb{S}^k \times \mathbb{S}^m)/\sim$ where $\sim$ is the equivalence relation generated by:

1. $(0, e_k, e_m) \sim (0, e_k, e'_m)$ for all $e_m, e'_m \in \mathbb{S}^m$
2. $(\frac{\pi}{2}, e_k, e_m) \sim (\frac{\pi}{2}, e'_k, e_m)$ for all $e_k, e'_k \in \mathbb{S}^k$.

Denote by $[\theta, e_k, e_m]$ the equivalence class of (θ, e_k, e_m). Let $e = [\theta, e_k, e_m]$ and $e' = [\theta', e'_k, e'_m]$. Define a metric d on $\mathbb{S}^k * \mathbb{S}^m$ by $d(e, e') \in [0, \pi]$ satisfying $\cos(d(e, e')) = \cos(\theta)\cos(\theta')\cos(d_k(e_k, e'_k)) + \sin(\theta)\sin(\theta')cos(d_m(e_m, e'_m))$. The join $\mathbb{S}^k * \mathbb{S}^m$ is isometric to $\mathbb{S}^{k+m+1}$; it contains an "arc" joining each $e_k \in \mathbb{S}^k$ to each $e_m \in \mathbb{S}^m$ with any two "arcs" intersecting at most at one endpoint.

We will need the following three lemmas for the proof of Theorem 3.8.

Lemma 3.1 *Let $e = [\theta, e_k, e_m] \in \mathbb{S}^k * \mathbb{S}^m$. Then $N_{\pi/2}(e) \subseteq N_{\pi/2}(e_k) \cup N_{\pi/2}(e_m)$ where $N_{\pi/2}(\cdot)$ is the $\frac{\pi}{2}$-neighborhood in $\mathbb{S}^k * \mathbb{S}^m$.*

PROOF: If $\theta = \frac{\pi}{2}$, then the lemma is obviously true, so suppose $\theta < \frac{\pi}{2}$. Let $e' = [\theta', e'_k, e'_m] \in N_{\pi/2}(e)$, and suppose e' is not in $N_{\pi/2}(e_m)$. We write $e_k = [0, e_k, e_m]$ and $e_m = [\frac{\pi}{2}, e_k, e_m]$. Since $d(e_m, e') \geq \frac{\pi}{2}$, we have $0 \geq \cos(d(e_m, e')) = \sin(\theta')\cos(d_m(e_m, e'_m))$ which implies that $\sin(\theta)\sin(\theta')\cos(d_m(e_m, e'_m)) \leq 0$ since $\sin(\theta) \geq 0$. Since $d(e, e') < \frac{\pi}{2}$, we have $0 < \cos(d(e, e')) = \cos(\theta)\cos(\theta')\cos(d_k(e_k, e'_k)) + \sin(\theta)\sin(\theta')\cos(d_m(e_m, e'_m))$. Therefore, $0 < \cos(\theta)\cos(\theta')\cos(d_k(e_k, e'_k))$ which implies $0 < \cos(\theta')\cos(d_k(e_k, e'_k)) = \cos(d(e_k, e'))$. Thus, $d(e_k, e') < \frac{\pi}{2}$. □

Lemma 3.2 *Suppose $e = [\theta, e_k, e_m] \in e_k * \mathbb{S}^m$ with $\theta < \frac{\pi}{2}$. Then $N^k_{\pi/2}(e_k) \subseteq N_{\pi/2}(e)$ where $N^k_{\pi/2}(\cdot)$ is the $\frac{\pi}{2}$-neighborhood in $\mathbb{S}^k$.*

PROOF: Let $e' = [0, e', e'_m] \in N^k_{\pi/2}(e_k)$. Since $d_k(e_k, e') < \frac{\pi}{2}$, we have $0 < \cos(d_k(e_k, e'))$ which implies $0 < \cos(\theta)\cos(d_k(e_k, e')) = \cos(d(e, e'))$. Therefore, $d(e, e') < \frac{\pi}{2}$. □

Lemma 3.3 *Suppose $e_k \in \mathbb{S}^k$. Then $N_{\pi/2}(e_k) = (N^k_{\pi/2}(e_k) * \mathbb{S}^m) - \mathbb{S}^m$.*

PROOF: Let $e_k = [0, e_k, e_m]$ for some $e_m \in \mathbb{S}^m$, and let $e' = [\theta', e'_k, e'_m]$ where $\theta' < \frac{\pi}{2}$. Then $e' \in N_{\pi/2}(e_k)$ if and only if $\cos(d(e_k, e')) > 0$ if and only if $\cos(\theta')\cos(d_k(e_k, e'_k)) > 0$ if and only if $\cos(d_k(e_k, e'_k)) > 0$ if and only if $(e' \in N^k_{\pi/2}(e_k) * \mathbb{S}^m) - \mathbb{S}^m$. □

3.2 Contractions toward $e \in \partial_\infty \mathbb{R}^m$

In this section, we review contractions toward $e \in \partial_\infty \mathbb{R}^m$ and some related ideas as discussed in [1, § 13.1]. Suppose $\rho : G \to \text{Transl}(\mathbb{R}^m)$ is an action of G on $\mathbb{R}^m$ by translations, X is an n-dimensional $(n-1)$-connected free G-CW complex with $G \backslash X$ finite, and $h : X \to \mathbb{R}^m$ is a G-map. Let $f : D(f) \to X$ be a cellular map where $D(f)$ is a subcomplex of X. A *shift of f toward $e \in \partial_\infty \mathbb{R}^m$* is a map $sh_{f,e} : D(f) \to \mathbb{R}$ defined by $sh_{f,e}(x) := \beta_\gamma \circ h \circ f(x) - \beta_\gamma \circ h(x)$ where γ is a geodesic ray representing e. The *guaranteed shift toward e* is $gsh_e(f) := \inf\{sh_{f,e}(x) | x \in D(f)\}$. A cellular map $\phi : X \to X$ is a *contraction toward e* if $gsh_e(\phi) > 0$. The following theorem relates contractions to $\Sigma^n(G)$.

Theorem 3.4 *[1, Theorem 14.5] There exists a contraction $\phi : X \to X$ toward e if and only if $e \in \Sigma^n(G)$.*

By combining Theorem 3.4 and Theorem 2.1, we get:

Corollary 3.5 *Let $e \in \partial_\infty \mathbb{R}^m$. Then $e \in \Omega^n(G)$ if and only if for every e' in an open $\frac{\pi}{2}$-neighborhood of e, there exists a contraction $\phi' : X \to X$ toward e'.*

Suppose G and H are groups of type F_n with $\text{Hom}(G, \mathbb{R}) \cong \mathbb{R}^k$ and $\text{Hom}(H, \mathbb{R}) \cong \mathbb{R}^m$. Then $\text{Hom}(G \times H, \mathbb{R}) \cong \mathbb{R}^{k+m}$. Let X_G (resp. X_H) be a contractible free G-CW complex (resp. H-CW complex) with $G \backslash X_G^n$ (resp. $H \backslash X_H^n$) finite, and let $h_G : X_G \to \mathbb{R}^k$ (resp. $h_H : X_H \to \mathbb{R}^m$) be a G-map (resp. H-map). Define $X := X_G \times X_H$ and $h := h_G \times h_H : X \to \mathbb{R}^{k+m}$. We will need the following lemmas for the proof of 3.8.

Lemma 3.6 *If there exists a contraction $\phi : X_G^n \to X_G^n$ toward $e \in \partial_\infty \mathbb{R}^k$, then $(\phi \times id_H) : X^n \to X^n$ is a contraction toward every $e' \in (e * \partial_\infty \mathbb{R}^m) - \partial_\infty \mathbb{R}^m$ where id_H is the identity map on X_H.*

PROOF: Let $(v, w) \in X^n$, let γ' be a geodesic ray defining $e' \in e * \partial_\infty \mathbb{R}^m$ with $\gamma'(0) = 0$, and let $u_{e'}$ be the unit vector (at 0) pointing toward e'. Let $p_k : \mathbb{R}^{k+m} \to \mathbb{R}^k$ and $p_m : \mathbb{R}^{k+m} \to \mathbb{R}^m$ be the natural projection maps.

$$
\begin{aligned}
sh_{(\phi \times id_H), e'}(v, w) &= \beta_{\gamma'} \circ h \circ (\phi \times id_H)(v, w) - \beta_{\gamma'} \circ h(v, w) \\
&= \beta_{\gamma'} \circ h(\phi(v), w) - \beta_{\gamma'} \circ h(v, w) \\
&= \beta_{\gamma'}(h_G \circ \phi(v), h_H(w)) - \beta_{\gamma'}(h_G(v), h_H(w)) \\
&= (u_{e'}, (h_G \circ \phi(v), h_H(w))) - (u_{e'}, (h_G(v), h_H(w))) \\
&= (p_k(u_{e'}), h_G \circ \phi(v)) + (p_m(u_{e'}), h_H(w)) - (p_k(u_{e'}), h_G(v)) - (p_m(u_{e'}), h_H(w)) \\
&= (p_k(u_{e'}), h_G \circ \phi(v)) - (p_k(u_{e'}), h_G(v)) \\
&= \|p_k(u_{e'})\| (\beta_{p_k \circ \gamma'} \circ h_G \circ \phi(v) - \beta_{p_k \circ \gamma'} \circ h_G(v)) \\
&\geq \|p_k(u_{e'})\| gsh_e(\phi).
\end{aligned}
$$

Thus, $gsh_{e'}(\phi \times id_H) > 0$ □

Lemma 3.7 *Let $e \in \partial_\infty \mathbb{R}^k \subseteq \partial_\infty \mathbb{R}^k * \partial_\infty \mathbb{R}^m$. Suppose there exists a contraction $\phi : X \to X$ toward e. Then there exists a contraction $\Phi : X_G \to X_G$ toward e.*

PROOF: Suppose $\phi : X \to X$ is a contraction toward $e \in \partial_\infty \mathbb{R}^k$. Let γ be a geodesic ray defining e, and let u_e be a vector (at 0) pointing toward e. Pick $x_H \in X_H$, and define $i : X_G \to X$ by $x_G \mapsto (x_G, x_H)$. Let $p_G : X \to X_G$ and $p_H : X \to X_H$ be the natural projection maps. Define $\Phi : X_G \to X_G$ by $\Phi(x_G) := p_G \circ \phi \circ i(x_G)$. Since u_e is pointing toward $e \in \partial_\infty \mathbb{R}^k$, its coordinates are $(a_1, \dots, a_k, 0, \dots, 0)$. Therefore, the function $\beta_\gamma : \mathbb{R}^{k+m} \to \mathbb{R}$ defined by $\beta_\gamma(a) := \langle u_e, a - \gamma(0)\rangle / \|u_e\|$ (used to define $sh_{\phi,e}$) restricted to $\mathbb{R}^k$ can be used to define $sh_{\Phi,e}$. Let $x_G \in X_G$.

$$\begin{aligned}
sh_{\Phi,e}(x_G) &= \beta_\gamma|_{\mathbb{R}^k} \circ h_G \circ \Phi(x_G) - \beta_\gamma|_{\mathbb{R}^k} \circ h_G(x_G) \\
&= \beta_\gamma|_{\mathbb{R}^k} \circ h_G \circ p_G \circ \phi \circ i(x_G) - \beta_\gamma|_{\mathbb{R}^k} \circ h_G(x_G) \\
&= \beta_\gamma|_{\mathbb{R}^k} \circ h_G \circ p_G \circ \phi(x_G, x_H) - \beta_\gamma|_{\mathbb{R}^k} \circ h_G(x_G) \\
&= \frac{\langle u_e, h_G \circ p_G \circ \phi(x_G, x_H)\rangle - \langle u_e, h_G(x_G)\rangle}{\|u_e\|}
\end{aligned}$$

Since u_e is pointing toward $e \in \partial_\infty \mathbb{R}^k$, we have

$$\begin{aligned}
&= \frac{\langle u_e, (h_G \circ p_G \circ \phi(x_G, x_H), h_H \circ p_H \circ \phi(x_G, x_H))\rangle}{\|u_e\|} - \frac{\langle u_e, (h_G(x_G), h_H(x_H))\rangle}{\|u_e\|} \\
&= \beta_\gamma(h_G \circ p_G \circ \phi(x_G, x_H), h_H \circ p_H \circ \phi(x_G, x_H)) - \beta_\gamma(h_G(x_G), h_H(x_H)) \\
&= \beta_\gamma \circ h \circ \phi(x_G, x_H) - \beta_\gamma \circ h(x_G, x_H) \\
&= sh_{\phi,e}(x_G, x_H) \\
&\geq gsh_e(\phi).
\end{aligned}$$

Thus, Φ is a contraction toward e. □

3.3 The Ω-invariant of a product

We are now ready to prove our main result.

Theorem 3.8 $\Omega^n(G \times H) = \Omega^n(G) * \Omega^n(H)$

PROOF: Let $e \in \Omega^n(G) * \Omega^n(H)$ and let $e = [\theta, e_G, e_H]$. Then $e_G \in \Omega^n(G)$ and $e_H \in \Omega^n(H)$. By Corollary 3.5, there is a contraction toward every $e'_G \in N^k_{\pi/2}(e_G)$ and every $e'_H \in N^m_{\pi/2}(e_H)$. By Lemma 3.6 and Lemma 3.3, there exists a contraction toward every element in $N^k_{\pi/2}(e_G) * \partial_\infty \mathbb{R}^m = N_{\pi/2}(e_G)$ and a contraction toward every element in $\partial_\infty \mathbb{R}^k * N^m_{\pi/2}(e_H) = N_{\pi/2}(e_H)$. By Lemma 3.1, $N_{\pi/2}(e) \subseteq N_{\pi/2}(e_G) \cup N_{\pi/2}(e_H)$, so there exists a contraction toward every $e' \in N_{\pi/2}(e)$. Thus, by Corollary 3.5, $e \in \Omega^n(G \times H)$.

For the reverse containment, let $e \in \Omega^n(G \times H)$. Then by Corollary 3.5, there exists a contraction toward every element in $N_{\pi/2}(e)$. By Lemma 3.2, $N^k_{\pi/2}(e_G) \subseteq N_{\pi/2}(e)$ and

$N^m_{\pi/2}(e_H) \subseteq N_{\pi/2}(e)$, so we have a contraction toward every element in $N^k_{\pi/2}(e_G)$ and every element in $N^m_{\pi/2}(e_H)$. By Lemma 3.7 and Corollary 3.5, $e_G \in \Omega^n(G)$ and $e_H \in \Omega^n(H)$, so $e \in \Omega^n(G) * \Omega^n(H)$. □

4 Bieri's Conjecture Implies The Main Result

We wish to show

Theorem 4.1 *Theorem 3.8 is a consequence of Conjecture 1.1.*

To do so, we need to fix some notation. For any $M \subseteq \mathbb{S}$, let $M^\perp := \{e \in \mathbb{S} | d(e, e_M) \geq \frac{\pi}{2}$ for all $e_M \in M\}$. Notice $N_{\pi/2}(M) = \mathbb{S} - M^\perp$; thus, Theorem 2.1 can be restated in this language as: $\Omega^n(G) = (\Sigma^n_c(G))^\perp$. We shall use $\perp$ in association with the metric d in $\partial_\infty \mathbb{R}^k * \partial_\infty \mathbb{R}^m$, $\perp_G$ in association with the metric d_G in $\partial_\infty \mathbb{R}^k$, and $\perp_H$ in association with the metric d_H in $\partial_\infty \mathbb{R}^m$. We will need the following two lemmas in the proof of Theorem 4.1.

Lemma 4.2 *Let $M_G \subseteq \mathbb{S}_G$ and $M_H \subseteq \mathbb{S}_H$. Then $(M_G * M_H)^\perp = M_G^\perp \cap M_H^\perp$.*

PROOF: Let $e \in (M_G * M_H)^\perp$. Thus, for all $e' \in M_G * M_H$, $d(e, e') \geq \frac{\pi}{2}$. Let $e_G \in M_G$. Since $M_G \subseteq M_G * M_H$, we have that $d(e, e_G) \geq \frac{\pi}{2}$, so $e \in M_G^\perp$. Similarly, $e \in M_H^\perp$, so $(M_G * M_H)^\perp \subseteq M_G^\perp \cap M_H^\perp$.

For the reverse containment, let $e \in M_G^\perp \cap M_H^\perp$, and let $e = [\theta, e_G, e_H]$. Let $e' \in M_G * M_H$ with $e' = [\theta', e'_G, e'_H]$. Since $e \in M_G^\perp$ and $e'_G \in M_G$, we have $0 \geq \cos(d(e, e'_G)) = \cos(\theta)\cos(d_G(e_G, e'_G))$. Similarly, $0 \geq \sin(\theta)\cos(d_H(e_H, e'_H))$. Thus, $\cos(d(e, e')) = \cos(\theta)$ $\cos(\theta')\cos(d_G(e_G, e'_G)) + \sin(\theta)\sin(\theta')\cos(d_H(e_H, e'_H)) \leq 0$, so $d(e, e') \geq \frac{\pi}{2}$. Therefore, we have $e \in (M_G * M_H)^\perp$. □

Lemma 4.3 *Let $M_G \subseteq \mathbb{S}_G$ and $M_H \subseteq \mathbb{S}_H$. Then $(M_G * M_H)^\perp = M_G^{\perp_G} * M_H^{\perp_H}$.*

PROOF: Let $e \in (M_G * M_H)^\perp$ with $e = [\theta, e_G, e_H]$. Let $e'_G \in M_G$ and $e'_H \in M_H$. Thus, $e'_G \in M_G * M_H$, so $d(e, e'_G) \geq \frac{\pi}{2}$. We have $0 \geq \cos(d(e, e'_G)) = \cos(\theta)\cos(d_G(e_G, e'_G))$ which implies $d_G(e_G, e'_G) \geq \frac{\pi}{2}$. Thus, $e_G \in M_G^{\perp_G}$. Similarly, $e_H \in M_H^{\perp_H}$, so $e \in M_G^{\perp_G} * M_H^{\perp_H}$.

For the reverse containment, let $e \in M_G^{\perp_G} * M_H^{\perp_H}$ with $e = [\theta, e_G, e_H]$. Let $e' \in M_G * M_H$ with $e' = [\theta', e'_G, e'_H]$. Since $e_G \in M_G^{\perp_G}$, we have $d_G(e_G, e'_G) \geq \frac{\pi}{2}$. Similarly, $d_H(e_H, e'_H) \geq \frac{\pi}{2}$. Thus, $\cos(d(e, e')) = \cos(\theta)\cos(\theta')\cos(d_G(e_G, e'_G)) + \sin(\theta)\sin(\theta')\cos(d_H(e_H, e'_H)) \leq 0$ which implies $e \in (M_G * M_H)^\perp$. □

PROOF OF THEOREM 4.1:

$$\begin{aligned}
\Omega^n(G) * \Omega^n(H) &= (\Sigma_c^n(G))^{\perp_G} * (\Sigma_c^n(H))^{\perp_H} && \text{by Theorem 2.1.}\\
&= (\Sigma_c^n(G))^{\perp} \cap (\Sigma_c^n(H))^{\perp} && \text{by Lemmas 4.2 and 4.3.}\\
&= \left(\bigcup_{i=0}^{n} \Sigma_c^i(G)\right)^{\perp} \cap \left(\bigcup_{i=0}^{n} \Sigma_c^{n-i}(H)\right)^{\perp} && \text{since } \Sigma_c^i(G) \subseteq \Sigma_c^n(G) \text{ for each } i \leq n.\\
&= \bigcap_{i=0}^{n} (\Sigma_c^i(G))^{\perp} \cap \bigcap_{i=0}^{n} (\Sigma_c^{n-i}(H))^{\perp}\\
&= \bigcap_{i=0}^{n} \left((\Sigma_c^i(G))^{\perp} \cap (\Sigma_c^{n-i}(H))^{\perp}\right)\\
&= \bigcap_{i=0}^{n} \left((\Sigma_c^i(G)) * (\Sigma_c^{n-i}(H))\right)^{\perp} && \text{by Lemma 4.2.}\\
&= \left(\bigcup_{i=0}^{n} (\Sigma_c^i(G)) * (\Sigma_c^{n-i}(H))\right)^{\perp}\\
&= (\Sigma_c^n(G \times H))^{\perp} && \text{by Conjecture 1.1.}\\
&= \Omega^n(G \times H) && \text{by Theorem 2.1.} \quad \square
\end{aligned}$$

5 Normal Subgroups of $G \times H$ with Abelian Quotient

In this section, we give a sufficient condition (in terms of $\Omega^n(G)$ and $\Omega^n(H)$) for a normal subgroup of $G \times H$ with abelian quotient to be of type F_n. First, we need a different description of $\partial_\infty \mathbb{R}^m$.

Define an equivalence relation on $\mathrm{Hom}(G, \mathbb{R})$ by: $\chi_1 \sim \chi_2$ if and only if $\chi_1 = r\chi_2$ for some $r > 0$. Denote by $[\chi]$ the equivalence class of χ, and define the *character sphere of G* to be $S(G) := \{[\chi] | 0 \neq \chi \in \mathrm{Hom}(G, \mathbb{R})\}$ with the quotient topology inherited from $\mathrm{Hom}(G, \mathbb{R})$. If $\mathrm{Hom}(G, \mathbb{R}) \cong \mathbb{R}^m$, then $S(G)$ is homeomorphic to $\partial_\infty \mathbb{R}^m$. For $N \leq G$, denote by $S(G, N) := \{[\chi] \in S(G) | N \leq ker(\chi)\}$. If N is normal with G/N abelian, then we have:

Theorem 5.1 *([3] for $n = 1$, [4] for $n \geq 2$): N is of type F_n if and only if $S(G, N) \subseteq \Sigma^n(G)$.*

Let $N \trianglelefteq G \times H$ with $G \times H/N$ abelian. Combining Theorem 2.1, Theorem 3.8, and Theorem 5.1, we get the following sufficient condition in terms of $\Omega^n(G)$ and $\Omega^n(H)$ for N to be F_n.

Corollary 5.2 *If $S(G \times H, N) \subseteq N_{\pi/2}(\Omega^n(G) * \Omega^n(H))$, then N is of type F_n.*

Example: We let $G \cong BS_{1,2} \cong \langle x, s | s^2 = x^{-1}sx\rangle$ and let $H \cong BS_{1,2} \cong \langle y, t | t^2 = y^{-1}ty\rangle$, so $G \times H \cong \langle x, s, y, t | s^2 = x^{-1}sx, t^2 = y^{-1}ty, [x, y] = 1, [x, t] = 1, [s, y] = 1, [s, t] = 1\rangle$. Thus, $\mathrm{Hom}(G \times H, \mathbb{R}) \cong \mathbb{R}^2$ with basis $x \mapsto (1, 0)$ and $y \mapsto (0, 1)$. The character sphere is $S(G \times H) \cong \mathbb{S}^0 * \mathbb{S}^0 \cong \mathbb{S}^1$. By Theorem 3.8, $\Omega^1(G \times H)$ is the closed upper right-hand quarter of $\mathbb{S}^1$ as shown in Figure 3. Denote by $\exp_x(g)$ the exponent sum of the x's in a word representing $g \in BS_{1,2} \times BS_{1,2}$, and denote by $\exp_y(g)$ the exponent sum of the y's in a word representing $g \in BS_{1,2} \times BS_{1,2}$. Note that $\exp_x(\cdot)$ (resp. $\exp_y(\cdot)$) is well-defined on $G \times H$ since it is well-defined on G (resp. H) (it is the "x-level" (resp. "y-level") in the Cayley graph in Figure 2). Let $N = \{g \in BS_{1,2} \times BS_{1,2} | \exp_x(g) = \exp_y(g)\}$. It is clear that N is a subgroup of $G \times H$ since for each $g_1, g_2 \in N$, we have $\exp_x(g_1 g_2^{-1}) = \exp_y(g_1 g_2^{-1})$. Since for each $g, h \in BS_{1,2} \times BS_{1,2}$, $[g, h] = g^{-1}h^{-1}gh \in N$, we have that $N \trianglelefteq G \times H$ and $G \times H/N$ is abelian. For $\phi \in \mathrm{Hom}(G \times H, \mathbb{R})$, $N \leq \ker(\phi)$ if and only if $\phi(x) = -\phi(y)$. The open $\frac{\pi}{2}$-neighborhood of $\Omega^1(G \times H)$ and $S(G \times H, N)$ are shown in Figure 3, and we can see by Corollary 5.2 that N is finitely generated.

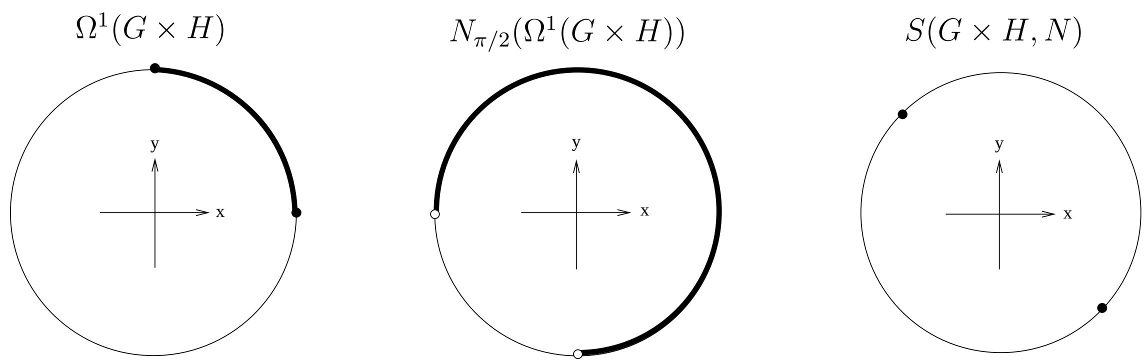

Figure 3: The bold portion of the left circle is $\Omega^1(BS_{1,2} \times BS_{1,2})$, the bold portion of the middle circle is $N_{\pi/2}(\Omega^1(BS_{1,2} \times BS_{1,2}))$, and the two points on the right circle are $S(BS_{1,2} \times BS_{1,2}, N)$.

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