Controlled Topology Invariants of Translation Actions

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Abstract

We develop invariants Ω^n of a translation action of a group on $\mathbb{R}^m$ analogous to the Bieri-Neumann-Strebel-Renz invariants Σ^n. The invariants Σ^n were defined to be the set of "directions" $e \in \partial_\infty \mathbb{R}^m$ such that a suitable universal G-space is $(n-1)$-connected over the half-spaces defined by e. We replace half-spaces by topologically more natural neighborhoods of e to obtain the new invariants Ω^n. The invariants Σ^n and Ω^n are related as follows: $e \in \Omega^n$ if and only if every e' in an open $\frac{\pi}{2}$-neighborhood of e lies in Σ^n.

2000 Mathematics Subject Classification: primary 20F65; secondary 57M07

Keywords: Σ-invariants, controlled $(n-1)$-connected, and bounded $(n-1)$-connected

1 Introduction

In this paper we develop new geometric invariants of a translation action ρ of a group G on $\mathbb{R}^m$. These are analogs of the Bieri-Neumann-Strebel-Renz (BNSR) invariants $\Sigma^n(\rho)$ introduced in [3] for $n = 1$ and in [4] for $n \geq 2$. For $n \geq 1$, the set $\Sigma^n(\rho)$ is a subset of the "$(m-1)$-sphere at infinity" of $\mathbb{R}^m$, denoted $\partial_\infty \mathbb{R}^m$ (ie the set of asymptotic equivalence classes of geodesic rays in $\mathbb{R}^m$). The description of $\Sigma^n(\rho)$ involves half-spaces of $\mathbb{R}^m$ perpendicular to the direction $e \in \partial_\infty \mathbb{R}^m$ as explained below. These half-spaces could be thought of as horoball neighborhoods of e. In controlled topology it is more natural to consider "ordinary" neighborhoods of e (in the topological m-ball $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$). This leads to a natural question: how does the description involving half-spaces compare to the description involving ordinary neighborhoods? In this paper we introduce new invariants $\Omega^n(\rho)$ having the same relation to ordinary neighborhoods of e as $\Sigma^n(\rho)$ has to horoball neighborhoods, and we describe the relationship between $\Omega^n(\rho)$ and $\Sigma^n(\rho)$.

1.1 The BNSR invariants Σ^n

Let n be a non-negative integer, let G be a group of type F_n, and let $\rho : G \to \text{Transl}(\mathbb{R}^m)$ be a cocompact action of G on $\mathbb{R}^m$ by translations[1]. There are two competing notions of neighborhood of $e \in \partial_\infty \mathbb{R}^m$ in $\mathbb{R}^m$:

[1] Denote by $\text{Transl}(\mathbb{R}^m)$ the group of translations of $\mathbb{R}^m$. For $g \in G$ and $a \in \mathbb{R}^m$, by ga we mean $\rho(g)(a)$.

1. ordinary neighborhoods of e in the compact space $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$ (intersected with $\mathbb{R}^m$).
2. half-spaces of $\mathbb{R}^m$ perpendicular to the direction e.

In [1], Bieri and Geoghegan defined the idea of ρ being *controlled* $(n-1)$*-connected* (or CC^{n-1}) in the direction e using notion (2) of neighborhoods. Here we recall the definition for $n = 1$; we give the full definition in § 2.1.

Let $(\cdot, \cdot)$ denote the Euclidean inner product, and for each $e \in \partial_\infty \mathbb{R}^m$, define $\beta_e : \mathbb{R}^m \to \mathbb{R}$ by $\beta_e(a) := (a, u_e)$, where u_e is the unit vector pointing in the direction e. For $s \in \mathbb{R}$, let $H_{e,s} := \beta_e^{-1}([s, \infty))$; $H_{e,s}$ is a closed half-space whose boundary is orthogonal to e. Let Γ be the Cayley graph of G with respect to a chosen finite generating set. Choose a G-map $h : \Gamma \to \mathbb{R}^m$ (for example, define $h(g) = \rho(g)(0)$ for all vertices $g \in G$ and extend linearly on all edges). Denote by $\Gamma_{e,s}$ the largest subgraph of Γ contained in $h^{-1}(H_{e,s})$. Say that ρ is CC^0 in the direction e if for each $s \in \mathbb{R}$, there exists $\lambda(s) \geq 0$ such that every $u, v \in \Gamma_{e,s}$ can be joined by a path in $\Gamma_{e,s-\lambda(s)}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$. The *Bieri-Neumann-Strebel invariant of* ρ is $\Sigma^1(\rho) := \{e \in \partial_\infty \mathbb{R}^m | \rho \text{ is } CC^0 \text{ in the direction } e\}$.

There is a generalization for $n \geq 2$, the *Bieri-Renz invariant of* ρ defined as $\Sigma^n(\rho) := \{e \in \partial_\infty \mathbb{R}^m | \rho \text{ is } CC^{n-1} \text{ in the direction } e\}$. We refer to all of these together as the *BNSR invariants.*

Some of the known theorems involving $\Sigma^n(\rho)$ are:

Theorem 1.1 *([3] for $n = 1$, [4] for $n \geq 2$) If ρ has discrete orbits, then the following are equivalent:*

1. *ρ is CC^{n-1} in all directions $e \in \partial_\infty \mathbb{R}^m$.*
2. *for each $a \in \mathbb{R}^m$, the stabilizer G_a has type F_n.*
3. *the kernel of the action ρ is type F_n.*

Theorem 1.2 *([3] for $n = 1$, [4] for $n \geq 2$) The set $\Sigma^n(\rho)$ is an open subset of $\partial_\infty \mathbb{R}^m$.*

1.2 The origins of Σ^n

Originally, Σ^n was an invariant of a group G. It is described in [2], and we recall that description. Let ρ be the canonical G-action on the real vector space $W := G/G' \otimes_{\mathbb{Z}} \mathbb{R}$. The vector space W has a natural base point 0. Denote by $\partial_\infty W$ the set of rays starting at 0. The set $\partial_\infty W$ is a sphere of dimension $(\dim W) - 1$. Choose an inner product $(\cdot, \cdot)$ for W. Then $\Sigma^n(G) := \Sigma^n(\rho)$.

1.3 The invariant Ω^n

From the point of view of topology, it is more natural to have a similar definition to CC^{n-1} using "ordinary" neighborhoods of e.

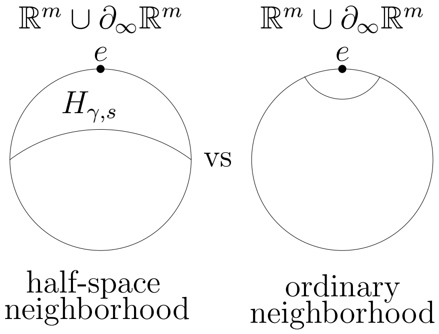

Figure 1: The figure on the left is the compactified space $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$ with half-space neighborhoods of e. The figure on the right is the compactified space $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$ with "ordinary" neighborhoods of e.

A basis for these neighborhoods consists of truncated cones. Let γ be the geodesic ray defining e with $\gamma(0) = 0$. For $s \geq 0$, the *truncated cone* $C_{\gamma,s} := Cone_\theta(\gamma) \cap H_{e,s}$ where $Cone_\theta(\gamma)$ is the closed cone of angle $\theta := \arctan(\frac{1}{s})$ (or $\theta = \frac{\pi}{2}$ if $s = 0$, in which case $C_{\gamma,s}$ is a half-space) with vertex $\gamma(0)$.

In § 2.2, we introduce the idea of ρ being *bounded* $(n-1)$*-connected* (or BC^{n-1}) in the direction e; we sketch it here for $n = 1$. Denote by $Y_{\gamma,s}$ the largest subgraph of Γ contained in $h^{-1}(C_{\gamma,s})$. Say that ρ is BC^0 in the direction e if there exists $s_0 \geq 0$ such that for each $s \geq s_0$, there exists $\lambda(s) \geq 0$ such that any two points $u, v \in Y_{\gamma,s}$ can be joined by a path in $Y_{\gamma,s-\lambda(s)}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$. The new invariant (replacing $\Sigma^1(\rho)$) is $\Omega^1(\rho) := \{e \in \partial_\infty \mathbb{R}^m | \rho$ is BC^0 in the direction $e\}$. As in the Σ-case, there is a generalization of this for $n \geq 2$, namely $\Omega^n(\rho) := \{e \in \partial_\infty \mathbb{R}^m | \rho$ is BC^{n-1} in the direction $e\}$.

The question of invariance of BC^{n-1} is answered in § 2.3. We see that ρ being BC^{n-1} in the direction e depends only on the action ρ and the direction e, so $\Omega^n(\rho)$ is an invariant of ρ. We compute Ω^n for right angled Artin groups in § 2.4. In § 3 we answer the following question: if ρ is BC^{n-1} in a direction, then is ρ CC^{n-1} in that same direction? and conversely? The complete answer is the following theorem:

Theorem 3.1. *The action ρ is BC^{n-1} in the direction $e \in \partial_\infty \mathbb{R}^m$ if and only if ρ is CC^{n-1} in all directions of an open $\frac{\pi}{2}$-neighborhood of e.*

Theorem 1.2 states that $\Sigma^n(\rho)$ is an open subset of $\partial_\infty \mathbb{R}^m$. As a Corollary to Theorem 3.1, we get that $\Omega^n(\rho)$ is a closed subset of $\partial_\infty \mathbb{R}^m$ (Corollary 3.13).

Given $\Sigma^n(\rho)$, we can completely determine $\Omega^n(\rho)$: for each $e \in \partial_\infty \mathbb{R}^m$, $e \in \Omega^n(\rho)$ if and only if the open $\frac{\pi}{2}$-neighborhood of e is in $\Sigma^n(\rho)$. However, it is not the case that $\Omega^n(\rho)$ completely determines $\Sigma^n(\rho)$. For each $e \in \partial_\infty \mathbb{R}^m$, if there exists $e' \in \Omega^n(\rho)$ in the open $\frac{\pi}{2}$-neighborhood[2] of e, then $e \in \Sigma^n(\rho)$, but if $\Omega^n(\rho) \cap N_{\pi/2}(e) = \emptyset$, this does not imply that

[2]Suppose X is a metric space and $\alpha > 0$. For each $x \in X$, denote by $N_\alpha(x) := \{y \in X | d(x,y) < \alpha\}$. Denote by $B_\alpha(x) := \{y \in X | d(x,y) \leq \alpha\}$. For each $A \subset X$, denote by $N_\alpha(A) := \{y \in X |$ there exists $z \in A$ such that $d(y,z) < \alpha\}$.

$e \in (\Sigma^n(\rho))^c$, the complement of $\Sigma^n(\rho)$. We use results in [5] to give an example of the latter.

For each $a \in \mathbb{Z}^m \subset \mathbb{R}^m$, denote by e_a the endpoint defined by the geodesic ray through a with initial point 0. The open hemisphere $N_{\pi/2}(e_a)$ is called *rationally defined.* A *rational convex polyhedral* subset of $\partial_\infty \mathbb{R}^m$ is the intersection of a finite number of rationally defined hemispheres; a *rational polyhedral* subset of $\partial_\infty \mathbb{R}^m$ is the union of a finite number of rational convex polyhedral subsets. The following is a Theorem of [5]:

Theorem 1.3 *Let P be a closed rational polyhedral subset of $\partial_\infty \mathbb{R}^m$. Then there exists a finitely presented group G such that $P = (\Sigma^1(G))^c$.*

Therefore, there exists a finitely presented group G_1 such that $\Sigma^1(G_1)$ is the union of the open $\frac{\pi}{2}$-neighborhood of the "north pole" e with the open $\frac{\pi}{4}$-neighborhood of the "south pole" $-e$ (see Figure 2). There also exists a finitely presented group G_2 such that $\Sigma^1(G_2)$ is only the open $\frac{\pi}{2}$-neighborhood of the "north pole" e. By Theorem 3.1, $\Omega^1(G_1) = \Omega^1(G_2) = e$, so we see that Ω^n cannot completely determine Σ^n.

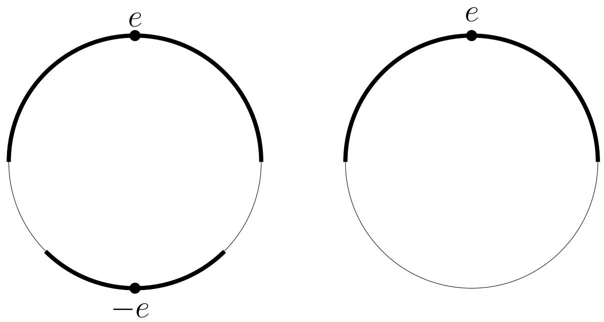

Figure 2: The figure on the left shows $\Sigma^1(G_1)$ in bold. It is the $\frac{\pi}{2}$-neighborhood of e and the $\frac{\pi}{4}$-neighborhood of $-e$. The figure on the right shows $\Sigma^1(G_2)$ in bold. It is the $\frac{\pi}{2}$-neighborhood of e. Thus, $\Omega^1(G_1) = \Omega^1(G_2) = e$.

This paper is based on the author's PhD dissertation written at SUNY-Binghamton under the direction of Ross Geoghegan.

2 Background, Definitions, and Some Observations

2.1 Controlled $(n-1)$-connected

Let $e \in \partial_\infty \mathbb{R}^m$, and let γ be a geodesic ray defining e. Associated to γ is the function $\beta_\gamma : \mathbb{R}^m \to \mathbb{R}$ defined by $\beta_\gamma(a) := (u_e, a - \gamma(0))$ where $(\cdot,\cdot)$ is the Euclidean inner product and u_e is the unit vector (at 0) pointing towards e. For each $s \in \mathbb{R}$, let $H_{\gamma,s} := \beta_\gamma^{-1}([s,\infty))$. Each $H_{\gamma,s}$ is a half-space of $\mathbb{R}^m$ whose boundary is perpendicular to γ.

Suppose that n is a non-negative integer, G is a group of type F_n, and $\rho : G \to \text{Transl}(\mathbb{R}^m)$ is an action of G on $\mathbb{R}^m$ by translations. Following Bieri and Geoghegan in [1], we define "controlled $(n-1)$-connected" as follows: Pick an n-dimensional $(n-1)$-connected CW complex X on which G acts freely as a group of cell permuting homeomorphisms with $G\backslash X$ a finite complex. Choose a G-map $h : X \to \mathbb{R}^m$ called a *control function*. Denote by $X_{\gamma,s}$ the largest subcomplex of X lying in $h^{-1}(H_{\gamma,s})$. The action ρ is *controlled* $(n-1)$-*connected* (or CC^{n-1}) *in the direction* e if for every $s \in \mathbb{R}$ and every $-1 \le p \le n-1$, there exists $\lambda = \lambda(s) \ge 0$ such that every map[3] $f : S^p \to X_{\gamma,s}$ can be extended to a map $\hat{f} : B^{p+1} \to X_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$. We call λ a *lag*. The *BNSR invariant* is $\Sigma^n(\rho) := \{e \in \partial_\infty \mathbb{R}^m | \rho \text{ is } CC^{n-1} \text{ in the direction } e\}$.

2.2 Bounded $(n-1)$-connected

In the compactified space $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$, the compactified half-spaces play the role of neighborhoods of the point $e \in \partial_\infty \mathbb{R}^m$ (see Figure 1), but this gives an unsatisfactory topology to $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$. A basis for a more natural topology consists of "truncated cones". Let $e \in \partial_\infty \mathbb{R}^m$, and let γ be a geodesic ray defining e. For each $s \ge 0$, the *truncated cone* (with respect to γ) is $C_{\gamma,s} := H_{\gamma,s} \cap Cone_\theta(\gamma)$ where:

1. $\theta := \arctan(\frac{1}{s})$ if $s > 0$ and $\theta := \frac{\pi}{2}$ if $s = 0$, and
2. $Cone_\theta(\gamma)$ is the closed cone of angle θ with vertex at $\gamma(0)$.

The *base* of the the truncated cone, denoted $base(C_{\gamma,s})$, is the $(m-1)$-ball of radius 1 and center $\gamma(s)$. Denote by $Y_{\gamma,s}$ the largest subcomplex of X contained in $h^{-1}(C_{\gamma,s})$. The action ρ is *bounded* $(n-1)$-*connected* (or BC^{n-1})*in the direction* e if there exists $r_0 \ge 0$ and there exists a function $\lambda : [r_0, \infty) \to [0, \infty)$ such that for each $r \ge r_0$ and each $-1 \le p \le n-1$, every map $f : S^p \to Y_{\gamma,r}$ can be extended to a map $\hat{f} : B^{p+1} \to Y_{\gamma,r-\lambda}$, and $r - \lambda(r) \to \infty$ as $r \to \infty$. The number r_0 in the definition will be referred to as the *initial base*. The new invariant is $\Omega^n(\rho) := \{e \in \partial_\infty \mathbb{R}^m | \rho \text{ is } BC^{n-1} \text{ in the direction } e\}$.

The following is an equivalent definition of BC^{n-1} which we provide for the reader's information.

Proposition 2.1 *An action ρ is BC^{n-1} in the direction e if and only if for each $s \ge 0$ and each $-1 \le p \le n-1$, there exists $t \ge s$ such that every map $f : S^p \to Y_{\gamma,t}$ can be extended to a map $\hat{f} : B^{p+1} \to Y_{\gamma,s}$.*

PROOF: For the "only if" direction, let $s \ge 0$. Define $t := \min\{r \in \mathbb{N} | r \ge r_0; r - \lambda(r) \ge s\}$. This set is non-empty since $r - \lambda(r) \to \infty$ as $r \to \infty$. Let $-1 \le p \le n-1$, and let $f : S^p \to Y_{\gamma,t}$ be a map. We have $C_{\gamma,t-\lambda(t)} \subseteq C_{\gamma,s}$ since $t - \lambda(t) \ge s$, so $Y_{\gamma,t-\lambda(t)} \subseteq Y_{\gamma,s}$. Thus, f can be extended to a map $\hat{f} : B^{p+1} \to Y_{\gamma,t-\lambda(t)} \subseteq Y_{\gamma,s}$.

For the "if" direction, let $\varepsilon > 0$. For each $s \ge 0$, define the set $T(s) := \{t \ge s|$ for each $-1 \le p \le n-1$, every map $f : S^p \to Y_{\gamma,t}$ extends to a map $\hat{f} : B^{p+1} \to Y_{\gamma,s}\}$.

[3]Throughout this paper, by a *map* we mean a continuous function

Define $t(s) := \inf(T(s)) + \varepsilon$, and define $r_0 := t(0)$. Let $r \geq r_0$ and $-1 \leq p \leq n-1$. There is a number $s \geq 0$ such that $t(s) \leq r$, so let $s_r := \sup\{s \geq 0 | t(s) \leq r\} - \varepsilon$. Let $\lambda(r) := r - s_r \geq 0$ and $f : S^p \to Y_{\gamma,r}$. Since $s_r < \sup\{s \geq 0 | t(s) \leq r\}$, there exists $s > s_r$ such that $t(s) \leq r$. Since $s > s_r$, we have $t(s) \geq t(s_r)$. Thus, $t(s_r) \leq r$. Therefore, $Y_{\gamma,r} \subseteq Y_{\gamma,t(s_r)}$, so $f : S^p \to Y_{\gamma,r} \subseteq Y_{\gamma,t(s_r)}$ which extends to a map $\hat{f} : B^{p+1} \to Y_{\gamma,s_r} = Y_{\gamma,r-\lambda}$. We are left to show $\lim_{r\to\infty} s_r = \infty$. Let $M > 0$. We need to find $r' \geq 0$ such that for every $r \geq r'$, $s_r \geq M$. Let $r' := t(M + \varepsilon)$ and let $r \geq r'$. $s_r + \varepsilon = \sup\{s > 0 | t(s) \leq r\}$, so $s_r + \varepsilon \geq M + \varepsilon$ which implies $s_r \geq M$. □

2.3 Invariance

In this section, we will show that our definition of BC^{n-1} depends only on ρ and e.

Proposition 2.2 *Let $h_1, h_2 : X \to \mathbb{R}^m$ be G-maps, and suppose ρ is BC^{n-1} in the direction e with respect to h_1. Then ρ is BC^{n-1} in the direction e with respect to h_2.*

PROOF: Since h_1 and h_2 are G-maps and $g \in G$ is an isometry, $d(h_1(gx), h_2(gx)) = d(gh_1(x), gh_2(x)) = d(h_1(x), h_2(x))$ for all $x \in X$. Since $G\backslash X$ is a finite CW complex (and thus, compact), $\alpha = \sup\{d(h_1(x), h_2(x)) | x \in X\}$ is finite. Let $s \geq 0$. There exists $s_0 \geq s$ such that $N_\alpha(C_{\gamma,s_0}) \subset C_{\gamma,s}$. We claim that $h_1^{-1}(C_{\gamma,s_0}) \subseteq h_2^{-1}(C_{\gamma,s})$. Indeed, if $y \in h_1^{-1}(C_{\gamma,s_0})$, then $h_1(y) \in C_{\gamma,s_0}$. We have $d(h_1(y), h_2(y)) \leq \alpha$, so $h_2(y) \in N_\alpha(C_{\gamma,s_0}) \subset C_{\gamma,s}$. Thus, $y \in h_2^{-1}(C_{\gamma,s})$ which proves the claim.

There exists $t_0 \geq s_0$ such that for each $-1 \leq p \leq n-1$, every map $f : S^p \to Y^1_{\gamma,t_0}$ extends to a map $\hat{f} : B^{p+1} \to Y^1_{\gamma,s_0}$ (here Y^1_{γ,t_0} will mean Y_{γ,t_0} with respect to h_1 and the superscript 2 will denote with respect to h_2). There exists $t \geq t_0$ such that $N_\alpha(C_{\gamma,t}) \subset C_{\gamma,t_0}$. For similar reasons as above, $h_2^{-1}(C_{\gamma,t}) \subseteq h_1^{-1}(C_{\gamma,t_0})$. Let $-1 \leq p \leq n-1$ and $f : S^p \to Y^2_{\gamma,t} \subseteq Y^1_{\gamma,t_0}$ be a map. f extends to $\hat{f} : B^{p+1} \to Y^1_{\gamma,s_0} \subseteq Y^2_{\gamma,s}$. Therefore, BC^{n-1} is independent of h. □

Lemma 2.3 *Let $\mu > 0$ and $r > 0$ with $r^2 > \mu^2 - 1$. Then there exists $\alpha = \alpha(r, \mu) > 0$ such that $N_\mu(C_{\gamma,r}) \subseteq C_{\gamma,r-\alpha}$, and for fixed μ, $r - \alpha(r) \to \infty$ as $r \to \infty$.*

PROOF: The required α is $\frac{(r^2+1)\mu}{\sqrt{r^2+1-\mu^2}+r\mu}$. □

Proposition 2.4 *Suppose ρ is BC^{n-1} in the direction e with respect to the geodesic ray γ and the G-map h, and suppose the initial base is r_0 and the lag is $\lambda : [r_0, \infty) \to [0, \infty)$. Suppose γ' defines e with $\|\gamma(0) - \gamma'(0)\| = \mu$. Then ρ is BC^{n-1} in the direction e with respect to γ' and h with initial base $r_0' = \inf\{r | r - \alpha(r, \mu) \geq r_0\}$ and lag $\lambda'(r) = \alpha(r, \mu) + \lambda(r - \alpha(r, \mu)) + \alpha(r - \alpha(r, \mu) - \lambda(r - \alpha(r, \mu)), \mu)$ where α is from Lemma 2.3.*

PROOF: There exists a translation τ of $\mathbb{R}^m$ such that $\gamma' = \tau \circ \gamma$. Let $\hat{h} = \tau \circ h$. For each $x \in X$, we have $\|h(x) - \hat{h}(x)\| = \mu$. Let $r_0' = \inf\{r | r - \alpha(r, \mu) > r_0\}$. Suppose $r \geq r_0'$ and

$f : S^p \to Y_{\gamma',r}$ is a map. If $y \in h^{-1}(C_{\gamma',r})$, then $\hat{h}(y) \in N_\mu(C_{\gamma',r}) \subseteq C_{\gamma',r-\alpha(r,\mu)}$. Therefore, $h^{-1}(C_{\gamma',r}) \subseteq \hat{h}^{-1}(C_{\gamma',r-\alpha(r,\mu)})$. Let $\hat{Y}_{\gamma',s}$ denote the largest subcomplex of X contained in $\hat{h}^{-1}(C_{\gamma',s})$. Thus, $f : S^p \to Y_{\gamma',r} \subseteq \hat{Y}_{\gamma',r-\alpha(r,\mu)}$ is a map, so there exists $\lambda = \lambda(r - \alpha(r,\mu)) \geq 0$ such that f extends to a map $\hat{f} : B^{p+1} \to \hat{Y}_{\gamma',r-\alpha(r,\mu)-\lambda}$. From Lemma 2.3, there exists $\alpha' = \alpha(r - \alpha(r,\mu) - \lambda, \mu) > 0$ such that $N_\mu(C_{\gamma',r-\alpha(r,\mu)-\lambda}) \subseteq C_{\gamma',r-\alpha(r,\mu)-\lambda-\alpha'}$. For similar reasons as above, $\hat{h}^{-1}(C_{\gamma',r-\alpha(r,\mu)-\lambda}) \subseteq h^{-1}(C_{\gamma',r-\alpha(r,\mu)-\lambda-\alpha'})$. Therefore, $\hat{f} : B^{p+1} \to \hat{Y}_{\gamma',r-\alpha(r,\mu)-\lambda} \subseteq Y_{\gamma',r-\alpha(r,\mu)-\lambda-\alpha'}$. We need only show that if $r \to \infty$, then $r - \lambda'(r) \to \infty$. We have that $r - \lambda'(r) = (r - \alpha(r,\mu)) - \lambda - \alpha'$. As $r \to \infty$, $(r - \alpha(r,\mu)) \to \infty$, so $((r - \alpha(r,\mu)) - \lambda \to \infty$. Thus, $r - \lambda'(r) \to \infty$. □

Proposition 2.5 *BC^{n-1} is independent of the choice of X.*

PROOF: The proof is the same as the proof of CC^{n-1} is independent of X in [1, Theorem 3.3]. □

2.4 Ω^n of right angled Artin groups

Let $\mathcal{G}$ be a finite simplicial graph with vertex set V. Associated to $\mathcal{G}$ is the *graph group* (or *right angled Artin group*) $G\mathcal{G}$ with presentation $\langle V | uv = vu$ for all $u, v \in V$ such that u and v are adjacent $\rangle$. Let $W := G\mathcal{G}/G\mathcal{G}' \otimes_{\mathbb{Z}} \mathbb{R}$, and give W an inner product $(\cdot,\cdot)$. Let ρ denote the canonical action of $G\mathcal{G}$ on W, so $\Sigma^n(G\mathcal{G}) := \Sigma^n(\rho)$ and $\Omega^n(G\mathcal{G}) := \Omega^n(\rho)$. The set $\{([v],1) \in W | v \in V\}$ forms a basis for W. We will abuse notation and refer to $v \in V$ as a basis element of W as well as a vertex of $\mathcal{G}$.

In [6], Meier, Meinert, and VanWyk give a complete computation of the Σ-invariants of $G\mathcal{G}$. We recall that result. Let $e \in \partial_\infty W$. A vertex v is *living* (with respect to e) if the directions v and e are perpendicular. Denote by $\mathcal{L}_e$ the full subgraph of $\mathcal{G}$ generated by the living vertices. The flag complex $\hat{\mathcal{G}}$ induced by $\mathcal{G}$ is the simplicial complex formed by filling in each complete subgraph of $\mathcal{G}$ with a simplex. Denote by $\hat{\mathcal{L}}_e$ the flag subcomplex of $\hat{\mathcal{G}}$ induced by $\mathcal{L}_e$. A subcomplex L of a simplicial complex K is (-1)-$\mathbb{Z}$-*acyclic-dominating* if it is non-empty. For $n \geq 0$, L is an n-$\mathbb{Z}$-*acyclic-dominating* subcomplex of K if for any vertex $v \in K - L$, the "restricted link" $lk_L(v) := lk(v) \cap L$ is $(n-1)$-acyclic and an $(n-1)$-$\mathbb{Z}$-acyclic-dominating subcomplex of the entire link $lk(v)$ in K.

Theorem 2.6 *(Meier,Meinert,VanWyk) $e \in \Sigma^n(G\mathcal{G})$ if and only if $\hat{\mathcal{L}}_e$ is $(n-1)$-connected and an $(n-1)$-$\mathbb{Z}$-acyclic-dominating subcomplex of $\hat{\mathcal{G}}$.*

We give a complete computation of the Ω-invariants of $G\mathcal{G}$ along with a proof that is independent of Theorem 2.6. A vertex $v \in V$ is *dominating* if v is adjacent to every other vertex in V. Suppose $U \subseteq V$. Denote by $\langle U \rangle$ the subspace of W spanned by U. Denote by $S(U)$ the subsphere $\partial_\infty \langle U \rangle$ of $\partial_\infty W$.

Theorem 2.7 *Let $\mathcal{G}$ be a finite simplicial graph, and let $G\mathcal{G}$ be the induced graph group. Let $U := \{v_1, \ldots, v_k\}$ be the set of dominating vertices of $\mathcal{G}$. Then $\Omega^n(G\mathcal{G}) = S(U)$ for all $n \geq 1$.*

We need a lemma.

Lemma 2.8 *If $\mathcal{G}$ has no dominating vertices, then $\Omega^n(G\mathcal{G}) = \emptyset$ for all $n \geq 1$.*

PROOF: It suffices to show that $\Omega^1(G\mathcal{G}) = \emptyset$. Let $e \in \partial_\infty W$, and let γ be the geodesic ray defining e with $\gamma(0) = 0$. Let X be the Cayley graph of $G\mathcal{G}$, and let $h : X \to W$ be a $G\mathcal{G}$-map. Pick $v \in V$ such that v is a "non-zero coordinate" of e.

Since $\mathcal{G}$ has no dominating vertices, there exists $u \in V$ and $s_0 > 0$ such that $uv \neq vu$ and $p(C_{\gamma,s_0}) \neq \langle\{u, v\}\rangle$ where $p : W \to \langle\{u, v\}\rangle$ is the natural projection map. The h-preimage of $\langle\{u, v\}\rangle$ is the Cayley graph of the free group on two generators F_2. Since $p \circ \gamma$ is a geodesic ray up to reparameteriztion, it defines a point $e' \in \partial_\infty\langle\{u, v\}\rangle$. Since $\Omega^1(F_2) = \emptyset$, there exists $s_1 \geq 0$ such that for every $t \geq s_1$, there exists $x, y \in Y_{p\circ\gamma,t}$ which cannot be joined in $Y_{p\circ\gamma,s_1}$. Although the projection of a truncated cone is not necessarily a truncated cone, there exists $s_2 \geq 0$ such that $p(C_{\gamma,s_2}) \subseteq C_{p\circ\gamma,s_1}$.

Let $t \geq s_2$. There exists $t_0 \geq s_1$ such that $C_{p\circ\gamma,t_0} \subseteq p(C_{\gamma,t})$. Therefore, there exists $x, y \in Y_{p\circ\gamma,t_0}$ which cannot be joined in $Y_{p\circ\gamma,s}$, so let $x_0, y_0 \in Y_{\gamma,t}$ be in the $\hat{p}$-preimage of x and y respectively where $\hat{p} : X \to X_U$ is the natural projection map with X_U denoting the subgraph of X generated by U. Then x_0 and y_0 cannot be joined by a path in Y_{γ,s_2} otherwise the projection of this path would join x and y in $Y_{p\circ\gamma,s_1}$. Thus, $e \in (\Omega^1(G\mathcal{G}))^c$. □

PROOF OF THEOREM 2.7: To show that $\Omega^n(G\mathcal{G}) \subseteq S(U)$, it suffices to show that $\Omega^1(G\mathcal{G}) \subseteq S(U)$. Suppose $e \in (S(U))^c$, and let γ be the geodesic ray defining e with $\gamma(0) = 0$. Let $\alpha > 0$ be the angle between γ and $\langle U\rangle$. There exists $s_0 > 0$ such that $\arctan(1/s_0) < \alpha$, so $C_{\gamma,s_0} \cap \langle U\rangle = \emptyset$. We have that $G\mathcal{G} \cong \mathbb{Z}^k \times H$ where H has presentation $\langle V - U|\mathcal{R}\rangle$ with $\mathcal{R}$ denoting the set of adjacency relations on $V - U$. Let $p_k : W \to \langle U\rangle$, $p_H : W \to \langle V - U\rangle$, and $\hat{p}_H : X \to X_{V-U}$ be the natural projection maps with X_{V-U} denoting the subgraph of X generated by $V - U$.

The graph associated to H has no dominating vertices, so by Lemma 2.8, we have $\Omega^1(H) = \emptyset$. The geodesic ray (up to reparameterization) $p_H \circ \gamma$ defines $e' \in \partial_\infty\langle V - U\rangle$. There exists $s_1 \geq 0$ such that for every $t \geq s_1$, there exists $x, y \in Y_{p_H\circ\gamma,t}$ which cannot be joined in $Y_{p_H\circ\gamma,s_1}$. There exists $s_2 \geq 0$ such that $p_H(C_{\gamma,s_2}) \subseteq C_{p_H\circ\gamma,s_1}$. Let $t \geq s_2$, so there exists $t_0 \geq s_1$ such that $C_{p_H\circ\gamma,t_0} \subseteq p_H(C_{\gamma,t})$. Therefore, there exists $x, y \in Y_{p_H\circ\gamma,t_0}$ which cannot be joined in $Y_{p_H\circ\gamma,s_1}$, so let $x_0, y_0 \in Y_{\gamma,t}$ be in the $\hat{p}_H$-preimage of x and y respectively. We have that x_0 and y_0 cannot be joined by a path in Y_{γ,s_2}, so $e \in (\Omega^1(G\mathcal{G}))^c$. Thus, $\Omega^n(G\mathcal{G}) \subseteq \Omega^1(G\mathcal{G}) \subseteq S(U)$.

We now show the reverse containment. Let $e \in S(U)$, and let γ be the geodesic ray defining e with $\gamma(0) = 0$. Since $G\mathcal{G} \cong \mathbb{Z}^k \times H$, we let X be $\mathbb{R}^k \times X(H)$ where $X(H)$ is the CW complex used for H acting on $\langle V - U\rangle$. Let $s \geq 0$, and let $f : S^p \to Y_{\gamma,s}$ be a map. There exists $F_H : B^{p+1} \to X(H)$ extending $p_H \circ f : S^p \to X(H)$. Since $p_H(Y_{\gamma,s}) = \langle V - U\rangle$, there exists $b \in \mathbb{R}^k$ such that $p_H((\{b\} \times X(H)) \cap Y_{\gamma,s}) \supseteq F_H(B^{p+1})$. Define $L : S^p \times I \to X = \mathbb{R}^k \times X(H)$ by $L(x, y) = ((1 - y)f(x) + yb, p_H \circ f(x))$. Since truncated cones are convex, we have $h \circ L$ is contained in $C_{\gamma,s}$, so the image of L is contained in $Y_{\gamma,s}$. We see that $L(x, 0) = f(x)$ and $L(x, 1) \subseteq (\{b\} \times X(H)) \cap Y_{\gamma,s}$. Thus, there is an extension of $L(S^p, 1)$ in $Y_{\gamma,s}$, and this extension concatenated with L gives the desired extension of f. $\square$

2.5 Equivalent definitions

In this section we establish some equivalent definitions of BC^{n-1} that will be used in proving the main theorem.

Lemma 2.9 *Suppose ρ is BC^{n-1} in the direction e with respect to γ with initial base r_0 and lag $\lambda : [r_0, \infty) \to [0, \infty)$. Then for each $g \in G$, ρ is BC^{n-1} in the direction e with respect to $g\gamma$ with the same initial base r_0 and lag $\lambda(r)$.*

PROOF: Let $g \in G$. Since $h : X \to \mathbb{R}^m$ is a G-map, $gh^{-1}(C_{\gamma,r_0}) = h^{-1}(C_{g\gamma,r_0})$. Let $r \geq r_0$ and $f : S^p \to Y_{g\gamma,r}$ be a map. Then $g^{-1} \circ f : S^p \to Y_{\gamma,r}$ extends to a map $\widehat{(g^{-1} \circ f)} : B^{p+1} \to Y_{\gamma,r-\lambda}$. The map $g\widehat{(g^{-1} \circ f)} : B^{p+1} \to Y_{g\gamma,r-\lambda(r)}$ is the desired extension. $\square$

Proposition 2.10 *Suppose ρ is a cocompact action on $\mathbb{R}^m$ by translations. Then ρ is BC^{n-1} in the direction e if and only if there exists $R_0 \geq 0$ and there exists $\Lambda : [R_0, \infty) \to [0, \infty)$ such that $r - \Lambda(r) \to \infty$ as $r \to \infty$, and for each $r \geq R_0$, for each $-1 \leq p \leq n-1$, and for every geodesic ray γ' defining e, every map $f : S^p \to Y_{\gamma',r}$ extends to a map $\hat{f} : B^{p+1} \to Y_{\gamma',r-\Lambda}$.*

PROOF: The "if" direction is obvious.

For the "only if" direction, suppose ρ is BC^{n-1} in the direction e with respect to γ with initial base r_0 and lag $\lambda : [r_0, \infty) \to [0, \infty)$. Define $R_0 := \sup\{r_0^{\gamma'} \geq 0 | r_0^{\gamma'}$ is the initial base with respect to γ'; $\gamma'(\infty) = e\}$. Since ρ is cocompact, there is a compact set $K \subset \mathbb{R}^m$ such that $\gamma(0) \in K$ and $\bigcup\{gK | g \in G\} = \mathbb{R}^m$. If γ' defines e, then there exists $g \in G$ such that $\gamma'(0) \in gK$. Suppose the diameter of K is D, so $r_0^{\gamma'} \leq \inf\{r | r - \alpha(r, D) > r_0\}$ by Proposition 2.4 and Lemma 2.9. Thus, $R_0 \leq \inf\{r | r - \alpha(r, D) > r_0\} < \infty$.

Let $r \geq R_0$ and $-1 \leq p \leq n-1$. Define $\Lambda(r) := \sup\{\lambda^{\gamma'}(r) \geq 0 | \lambda^{\gamma'}(r)$ is the lag with respect to γ'; $\gamma'(0) = e\}$. By Proposition 2.4 and Lemma 2.9, $\lambda^{\gamma'}(r) \leq \alpha(r, D) + \lambda(r - \alpha(r, D)) + \alpha(r - \alpha(r, D) - \lambda(r - \alpha(r, D)), D)$. Thus, $\Lambda(r) < \infty$, and as $r \to \infty$, $r - \Lambda(r) \to \infty$.

Let γ' define e, and let $f : S^p \to Y_{\gamma',r}$ be a map. Since $r \geq R_0 \geq r_0^{\gamma'}$, there exists $\lambda^{\gamma'}(r) \geq 0$ such that $\hat{f} : B^{p+1} \to Y_{\gamma',r-\lambda^{\gamma'}}$ extends f. Since $\Lambda(r) \geq \lambda^{\gamma'}(r)$, we have $r - \Lambda(r) \leq r - \lambda^{\gamma'}(r)$, so $Y_{\gamma',r-\lambda^{\gamma'}} \subseteq Y_{\gamma',r-\Lambda}$. Therefore, $\hat{f} : B^{p+1} \to Y_{\gamma',r-\Lambda}$ extends f. $\square$

We will later need the BC^{n-1} definition for the preimage of open truncated cones. Denote by $\overset{\circ}{Y}_{\gamma,r}$ the largest subcomplex contained in $h^{-1}(\text{int}(C_{\gamma,r}))$.

Proposition 2.11 *Suppose ρ is a cocompact action. Then ρ is BC^{n-1} in the direction e if and only if there exists $r_0 \geq 0$ and $\lambda : [r_0, \infty) \to [0, \infty)$ such that $r - \lambda(r) \to \infty$ as $r \to \infty$,* *and for each $r \geq r_0$, for each $-1 \leq p \leq n-1$, and for every geodesic ray γ defining e, every map $f : S^p \to \overset{\circ}{Y}_{\gamma,r}$ extends to a map $\hat{f} : B^{p+1} \to \overset{\circ}{Y}_{\gamma,r-\lambda}$.*

PROOF: For the "only if" direction, suppose ρ is BC^{n-1} in the direction e. Let $\varepsilon > 0$. Let $r_0 = R_0$ where R_0 is as in Proposition 2.10, and suppose $r \geq r_0$ and $-1 \leq p \leq n-1$. Then $\lambda(r) := \Lambda(r) + \varepsilon$ where $\Lambda(r)$ is as in Proposition 2.10.

For the "if" direction, let $\varepsilon > 0$. Let $r \geq r_0$ and $-1 \leq p \leq n-1$. Define $\lambda'(r) := \lambda(r) + \varepsilon$. Suppose γ defines e and $f : S^p \to Y_{\gamma,r}$ is a map. Let γ_ε be the geodesic ray defining e so that $\gamma_\varepsilon(\varepsilon) = \gamma(0)$. Therefore, $C_{\gamma,r} \subset \text{int}(C_{\gamma_\varepsilon,r})$ and $\text{int}(C_{\gamma_\varepsilon,r-\lambda}) \subset C_{\gamma,r-\lambda'}$. □

3 The Main Theorem

In this section, we prove our main theorem.

Theorem 3.1 *The action ρ is BC^{n-1} in the direction e if and only if ρ is CC^{n-1} in all directions in an open $\frac{\pi}{2}$-neighborhood of e.*

In preparation for the "only if" direction (Theorem 3.9), we need to discuss μ-selections, and for the "if" direction (Theorem 3.10), we need to discuss sheaves of maps.

3.1 μ-Selections

This section follows the work of E. Michael in [8]. Michael worked with selections (or "0-selections"), but since we allow lag in our definition of BC^{n-1}, we need to use "μ-selections" where $\mu > 0$. We will obtain a "μ-Selection Theorem" analogous to Michael's "Selection Theorem".

Let B be a topological space and X be a metric space. Denote by $2^X - \emptyset$ the set of all non-empty subsets of X. Given $\varphi : B \to (2^X - \emptyset)$ and $\mu > 0$, a map $f : B \to X$ is a *μ-selection* for φ if $f(t) \in N_\mu(\varphi(t))$ for each $t \in B$. A 0-selection is what Michael calls a *selection*.

The space X is *uniformly LC^{n-1}* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $m \leq n-1$, every singular S^m in X of diameter $\leq \delta$ bounds a singular B^{m+1} in X of diameter $< \varepsilon$.

Suppose we are given $\mathcal{S}_0, \ldots, \mathcal{S}_\ell \subseteq (2^X - \emptyset)$ and for each $0 \leq i \leq \ell - 1$ a bijection $\Psi_i : \mathcal{S}_i \to \mathcal{S}_{i+1}$ such that for every $S \in \mathcal{S}_i$, $S \subseteq \Psi_i(S)$. Let $\mathcal{S} := \{(S_0, \ldots, S_\ell) \in \mathcal{S}_0 \times \ldots \times \mathcal{S}_\ell |$ for each $0 \leq i \leq \ell - 1, S_{i+1} = \Psi_i(S_i)\}$. An $(\ell+1)$-tuple $(S_0, \ldots, S_\ell) \in \mathcal{S}$ is *uniformly LC^{n-1}* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $0 \leq i \leq \ell - 1$ and each $m \leq n-1$, every singular S^m in S_i of diameter $\leq \delta$ bounds a singular B^{m+1} in S_{i+1} of diameter $< \varepsilon$. The set $\mathcal{S}$ is *uniformly equi-LC^{n-1}* if each $(S_0, \ldots, S_\ell) \in \mathcal{S}$ is uniformly LC^{n-1} with uniform $\delta > 0$. An $(\ell+1)$-tuple $(S_0, \ldots, S_\ell) \in \mathcal{S}$ is C^{n-1} if for each $0 \leq i \leq \ell - 1$ and each $m \leq n-1$, every singular S^m in S_i bounds a singular B^{m+1} in S_{i+1}. A function $\varphi_i : B \to \mathcal{S}_i$ is *lower semi-continuous* (or l.s.c.) if for every open $V \subseteq X$, the set $\{t \in B | \varphi_i(t) \cap V \neq \emptyset\}$ is open in B.

Let n be a positive integer, and let $\ell = n^2$. Suppose $\varphi_0 : B^n \to \mathcal{S}_0$ is given, and for each $1 \leq i \leq \ell$, $\varphi_i : B^n \to \mathcal{S}_i$ is defined by $\varphi_i := \Psi_{i-1} \circ \varphi_{i-1}$. We assume:

1. X is an n-dimensional, $(n-1)$-connected, uniformly LC^{n-1} metric space.
2. $\mathcal{S}$ is uniformly equi-LC^{n-1}.
3. For each $0 \leq i \leq \ell$, $\varphi_i : B^n \to \mathcal{S}_i$ is l.s.c.
4. Each $(S_0, \ldots, S_\ell) \in \mathcal{S}$ is C^{n-1}.

Our μ-Selection Theorem is:

Theorem 3.2 *Under these hypotheses, let $g : S^{n-1} \to X$ be a selection for $\varphi_0 | S^{n-1}$. For every $\mu > 0$, there exists a μ-selection $f : B^n \to X$ for φ_ℓ such that $f(t) \in N_\mu(g(t))$ for every $t \in S^{n-1}$.*

To prove Theorem 3.2, we will give a series of seemingly weaker conditions culminating in the proofs of Propositions 3.4 and 3.5. We start with the following proposition[4].

Proposition 3.3 *Under these hypotheses, for every $\mu > 0$, there exists a triangulation K of B^n and a map $f : K \to X$ such that if σ is a simplex of K and $t \in \mathrm{st}(\sigma)$, then $f(\sigma) \subseteq N_\mu(\varphi_\ell(t))$.*

PROOF THAT PROPOSITION 3.3 $\Rightarrow$ THEOREM 3.2: Let $\mu > 0$. Define $\mathcal{S}' := \mathcal{S} \cup \{(x, \ldots, x) | x \in S_0 \in \mathcal{S}_0\}$. For each $0 \leq i \leq \ell$, define $\psi_i : B^n \to \mathcal{S}_i$ by $\psi_i(t) = g(t)$ if $t \in S^{n-1}$ and $\psi_i(t) = \varphi_i(t)$ otherwise. We check that $\mathcal{S}'$ and each ψ_i satisfy (1)-(4).

Condition (1) is satisfied. For condition (2), since $\mathcal{S}$ is uniformly equi-LC^{n-1}, we have $\mathcal{S}'$ is uniformly equi-LC^{n-1} with the same $\delta(\varepsilon)$ as for $\mathcal{S}$. Condition (3) is satisfied by [7, Example 1.3*]. For condition (4), since each $(S_0, \ldots, S_\ell) \in \mathcal{S}$ is C^{n-1} and each $(x, \ldots, x)$ is C^{n-1}, each $(S'_0, \ldots, S'_\ell) \in \mathcal{S}'$ is C^{n-1}.

By Proposition 3.3, there is a triangulation K of B^n and a map $f : K \to X$ such that if σ is a simplex of K and $t \in \mathrm{st}(\sigma)$, then $f(\sigma) \subseteq N_\mu(\psi_\ell(t))$. Let $t \in B^n$, so $t \in \mathring{\sigma}$ for some simplex σ which implies $t \in \mathrm{st}(\sigma)$. Thus, $f(t) \in f(\sigma) \subseteq N_\mu(\psi_\ell(t)) \subseteq N_\mu(\varphi_\ell(t))$, so f is a μ-selection for φ_ℓ. If $t \in S^{n-1}$, then $f(t) \in N_\mu(\psi_\ell(t)) = N_\mu(g(t))$. □

Proposition 3.4 *Assume Condition (3). For every $\alpha > 0$, there exists a triangulation K of B^n and a map $f : K^0 \to X$ such that if $v \in K^0$ and $t \in \mathrm{st}(v)$, then $f(v) \in N_\alpha(\varphi_0(t))$.*

[4] For each simplex σ of a simplicial complex, the *star of* σ, denoted $\mathrm{St}(\sigma)$, is the subcomplex generated by all the simplices containing σ. The *open star of* σ, denoted $\mathrm{st}(\sigma)$, is the interior of $\mathrm{St}(\sigma)$.

It will be useful to have the following definition. Let $\nu > 0$, let $0 \leq i \leq n-1$, let K be a triangulation of B^n, and let $f : K^i \to X$ be a map. Then K and f have *type* $\langle \nu, i \rangle$ if whenever σ is a simplex of K and $t \in \text{st}(\sigma)$, then $f(\sigma \cap K^i) \subseteq N_\nu(\varphi_{in}(t))$.

Proposition 3.5 *Under the same hypotheses, for every $\mu > 0$, there exists $\alpha(\mu) > 0$ such that if for each $0 \leq i \leq n-1$, the triangulation K of B^n and the map $f : K^i \to X$ have type $\langle \alpha, i \rangle$, then there exists a triangulation L of B^n and a map $g : L^{i+1} \to X$ having type $\langle \mu, i+1 \rangle$.*

PROOF THAT PROPOSITIONS 3.4 AND 3.5 $\Rightarrow$ PROPOSITION 3.3: Let $\mu > 0$. Assume without loss of generality that $\alpha(\mu) \leq \mu$ for all $\mu > 0$ in Proposition 3.5. Let $\alpha^{(0)}(\mu) := \mu$ and $\alpha^{(k+1)}(\mu) := \alpha(\alpha^{(k)}(\mu))$. By Proposition 3.4, there is a triangulation K_0 of B^n and a map $f_0 : (K_0)^0 \to X$ such that if $v \in (K_0)^0$ and $t \in \text{st}(v)$, then $f_0(v) \in N_{\alpha^{(n)}}(\varphi_0(t))$. Suppose σ is a simplex of K_0 and $t \in \text{st}(\sigma)$, we have $f_0(\sigma \cap (K_0)^0) \subseteq N_{\alpha^{(n)}}(\varphi_0(t))$ since $t \in \text{st}(\sigma)$ implies $t \in \text{st}(v)$ for each vertex v of σ.

By induction, for each $0 \leq i \leq n$, there is a triangulation K_i of B^n and a map $f_i : (K_i)^i \to X$ such that whenever σ is a simplex of K_i and $t \in \text{st}(\sigma)$, then $f_i(\sigma \cap (K_i)^i) \subseteq N_{\alpha^{(n-i)}}(\varphi_{in}(t))$. Let $K := K_n$ and $f := f_n$. Suppose σ is a simplex of K and $t \in \text{st}(\sigma)$, then $f(\sigma) = f(\sigma \cap K) \subseteq N_\mu(\varphi_\ell(t))$. □

PROOF OF PROPOSITION 3.4: Given Condition (3) and $\alpha > 0$. For each $t \in B^n$, pick $x_t \in \varphi_0(t)$. Let $U_t := \{t' \in B^n | \varphi_0(t') \cap N_\alpha(x_t) \neq \emptyset\}$. Since φ_0 is l.s.c., U_t is open in B^n, so $\mathcal{U} := \{U_t | t \in B^n\}$ is an open cover of B^n. Therefore, B^n has a triangulation K that is finer[5] than $\mathcal{U}$. For each vertex v, pick $t_v \in B^n$ so that $\text{st}(v) \subseteq U_{t_v}$, and let $f(v) := x_{t_v}$. Let $v \in K^0$ and $t' \in \text{st}(v)$, so $t' \in U_{t_v}$. Therefore, $\varphi_0(t') \cap N_\alpha(x_{t_v}) \neq \emptyset$, so $x_{t_v} = f(v) \in N_\alpha(\varphi_0(t'))$. □

Before we prove Proposition 3.5, we need three lemmas.

Lemma 3.6 *Suppose X is uniformly LC^{n-1} and $g : S^{n-1} \to X$ is a map. For every $\mu > 0$, there exists $\nu(\mu) > 0$ such that for every map $f : S^{n-1} \to X$ with $f(t) \in N_\nu(g(t))$ for all $t \in S^{n-1}$, there exists a homotopy $H : S^{n-1} \times I \to X$ from g to f such that $H(t,s) \in N_\mu(g(t))$ for each $t \in S^{n-1}$ and each $s \in I$.*

PROOF: This holds because X is uniformly LC^{n-1}. □

Lemma 3.7 *Given Condition (1). Let $(S_i, \ldots, S_{n+i-1})$ be uniformly LC^{n-1}. Then for every $\gamma > 0$, there exists $\kappa(\gamma) > 0$ such that if $k : S^m \to N_\kappa(S_i) (m \leq n-1)$ is a map, then there exists a map $f : S^m \to S_{n+i-1}$ such that $f(t) \in N_\gamma(k(t))$ for each $t \in S^m$.*

PROOF: By induction, for every $\gamma > 0$, there exists $\nu(\gamma) > 0$ such that if K is a simplicial complex of dimension $\leq n-1$ and $u : K^0 \to S_i$ is such that $\text{diam}(u(\sigma \cap K^0)) < \nu$ for each

[5] A triangulation K is *finer* than an open cover $\mathcal{U}$ if for each vertex v of K, there exists $U \in \mathcal{U}$ such that $\text{st}(v) \subseteq U$

simplex σ of K, then u can be extended to a map $f : K \to S_{n+i-1}$ with $\mathrm{diam}(f(\sigma)) < \gamma$. Without loss of generality, assume $\nu(\gamma) \leq \gamma$. Let $\kappa(\gamma) := \frac{1}{4}\nu(\frac{1}{3}\gamma)$.

Cover S^m by open sets whose images under k have diameter $< \kappa$. Then S^m has a triangulation L that is finer than this cover, so $\mathrm{diam}(k(\mathrm{st}(v))) < \kappa$ for each vertex v. For each $v \in L^0$, pick $t_v \in \mathrm{st}(v)$ and $u(v) \in S_i$ so that $d(u(v), k(t_v)) < \kappa$.

Suppose $v_1, v_2 \in L^0$ with v_1 and v_2 are joined by an edge. Then $d(u(v_1), u(v_2)) \leq d(u(v_1), k(t_{v_1})) + d(k(t_{v_1}), k(t_{v_2})) + d(k(t_{v_2}), u(v_2)) \leq \kappa + 2\kappa + \kappa = \nu(\frac{1}{3}\gamma)$. Thus, there is a map $f : L \to S_{n+i-1}$ with $\mathrm{diam}(f(\sigma)) < \frac{1}{3}\gamma$. Let $t \in S^m$, so $t \in \mathrm{st}(v)$ for some $v \in L^0$. Therefore, $d(f(t), k(t)) \leq d(f(t), f(v)) + d(f(v), k(t_v)) + d(k(t_v), k(t)) < \frac{1}{3}\gamma + \kappa + \kappa \leq \gamma$. □

Lemma 3.8 *Given Condition (1). Let $(S_i, \ldots, S_{n+i})$ be uniformly LC^{n-1} and C^{n-1}. Then for every $\mu > 0$, there exists $\alpha(\mu) > 0$ such that for each $m \leq n-1$, every map $k : S^m \to N_\alpha(S_i)$ is homotopic in $N_\mu(S_{n+i})$ to a constant map.*

PROOF: Let κ be as in Lemma 3.7 and ν be as in Lemma 3.6. Without loss of generality, assume $\nu(\mu) \leq \mu$ for all $\mu > 0$. Let $\alpha(\mu) := \kappa(\nu(\mu))$, and let $k : S^m \to N_\alpha(S_i)$ be a map. By Lemma 3.7, there exists a map $f : S^m \to S_{n+i-1}$ such that $f(t) \in N_{\nu(\mu)}(k(t))$ for all $t \in S^m$. By Lemma 3.6, there exists a homotopy h_1 between k and f such that the image of h_1 is contained in $N_\mu(f(S^m))$. Since $(S_i, \ldots, S_{n+i})$ is C^{n-1}, f is homotopic via a map h_2 to a constant map over a subset of S_{n+i}. Combining h_1 and h_2, we get a homotopy h between k and a constant map over a subset of $N_\mu(S_{n+i})$. □

PROOF OF PROPOSITION 3.5: Let $\alpha(\mu)$ be as in Lemma 3.8. Without loss of generality, assume $\alpha(\mu) \leq \mu$ for all $\mu > 0$. Suppose $0 \leq i \leq n-1$, and suppose K and f have type $\langle \alpha, i \rangle$.

For each $t \in B^n$ and each simplex σ of K^{i+1} with $t \in \mathrm{st}(\sigma)$, define $f_{t,\sigma} : \sigma \to X$ by:

1. if $\dim(\sigma) \leq i$, then $f_{t,\sigma} := f|_\sigma$.
2. if $\dim(\sigma) = i+1$, then $f_{t,\sigma}$ is a continuous extension of $f|_{\dot{\sigma}}$ such that $f_{t,\sigma}(\sigma) \subseteq N_\mu(\varphi_{(i+1)n}(t))$ (this is possible by Lemma 3.8).

Let $W_{t,\sigma} := \{t' \in B^n | f_{t,\sigma}(\sigma) \subseteq N_\mu(\varphi_{(i+1)n}(t'))\}$. Thus, $t \in W_{t,\sigma}$, and by [8, Lemma 11.3], $W_{t,\sigma}$ is open in B^n. Let $W_t := \bigcap\{W_{t,\sigma} | t \in \mathrm{st}(\sigma)\}$, let $M_t := \bigcap\{\ \mathrm{st}(u) | t \in \mathrm{st}(u)\}$, and let $R_t := W_t \cap M_t$. By [8, Lemma 11.4], for each $t \in B^n$, there is a neighborhood $T_t \subseteq R_t$ of t and there exists $t' \in B^n$ such that if $\bigcap T_{t_\beta} \neq \emptyset$, then $\bigcup T_{t_\beta} \subseteq \bigcap R_{t'_\beta}$. The cover $\mathcal{U} := \{T_t | t \in B^n\}$ is an open cover of B^n, so B^n has a triangulation L finer than $\mathcal{U}$.

For each $v \in L^0$, pick $t_v \in B^n$ such that $\mathrm{st}(v) \subseteq T_{t_v}$, and pick $u_v \in K^0$ such that $t'_v \in \mathrm{st}(u_v)$. Therefore, $\mathrm{st}(v) \subseteq T_{t_v} \subseteq R_{t'_v} \subseteq M_{t'_v} \subseteq \mathrm{st}(u_v)$, so we have a map $v \mapsto u_v$. This map can be extended naturally to a simplicial map $\pi : L^i \to K^i$ since given τ with vertices $v_1, \ldots, v_k$, we have $\bigcap\{\mathrm{st}(v_j) | 1 \leq j \leq k\} \neq \emptyset$. Thus, $\bigcap\{\mathrm{st}(u_{v_j}) | 1 \leq j \leq k\} \neq \emptyset$ which means that $u_{v_1}, \ldots, u_{v_k}$ are the vertices of a simplex. Define $\omega := f \circ \pi$.

We will define g as an extension of ω to L^{i+1}. To do so, we need only consider $(i+1)$-simplices of L. Let σ be an $(i+1)$-simplex of L with vertices $v_1, \ldots, v_{i+2}$ and denote t_{v_1} by t_σ. Then $\bigcap\{T_{v_j} | 1 \le j \le i+2\} \supseteq \bigcap\{\text{st}(v_j) | 1 \le j \le i+2\} \ne \emptyset$, so $t_\sigma \in T_{t_\sigma} \subseteq \bigcap\{R_{t'_{v_j}} | 1 \le j \le i+2\} \subseteq \bigcap\{\text{st}(u_{v_j}) | 1 \le j \le i+2\}$. Therefore, t_σ is in the intersection of the stars of the vertices of $\pi(\sigma)$, so $t_\sigma \in \text{st}(\pi(\sigma))$. We extend $\omega|_{\mathring{\sigma}}$ to $g_\sigma : \sigma \to X$ by $g_\sigma := f_{t_\sigma, \pi(\sigma)} \circ \pi$, so $g : L^{i+1} \to X$ is defined by $g|_\sigma = g_\sigma$.

We need to show if τ is a simplex of L and $t' \in \text{st}(\tau)$, then $g(\tau \cap L^{i+1}) \subseteq N_\mu(\varphi_{(i+1)n}(t'))$. To do this, it suffices to show that if σ is a simplex of L^{i+1} and $t \in \text{st}(\sigma)$, then $g(\sigma) \subseteq N_\mu(\varphi_{(i+1)n}(t))$. Suppose σ has vertices $v_1, \ldots, v_k$, so $t \in \bigcap\{\text{st}(v_j) | 1 \le j \le k\} \subseteq \bigcap\{\text{st}(u_{v_j}) | 1 \le j \le k\}$. If $\dim(\sigma) \le i$, then $\dim(\pi(\sigma)) \le i$, so $g(\sigma) := f_{t_\sigma, \pi(\sigma)} \circ \pi(\sigma) = f \circ \pi(\sigma) \subseteq N_\alpha(\varphi_{in}(t)) \subseteq N_\mu(\varphi_{(i+1)n}(t))$. Now suppose $\dim(\sigma) = i+1$. Since $t \in \text{st}(\sigma)$, we have $t \in \text{st}(v_j)$ for each $1 \le j \le k$, so $t \in \text{st}(v_1) \subseteq T_{t_\sigma} \subseteq R_{t_\sigma} \subseteq W_{t_\sigma} \subseteq W_{t_\sigma, \pi(\sigma)}$ with the last inclusion holding since $t_\sigma \in \text{st}(\pi(\sigma))$ from above. Thus, $g(\sigma) = f_{t_\sigma, \pi(\sigma)} \circ \pi(\sigma) \subseteq N_\mu(\varphi_{(i+1)n}(t))$. $\square$

3.2 The "only if" direction

We have an n-dimensional, $(n-1)$-connected, free G-CW complex Y with $G\backslash Y$ finite, but we wish to use a simplicial complex instead of a CW complex. It is well-known that given Y, there is an n-dimensional, $(n-1)$-connected, free G-simplicial complex X with $G\backslash X$ finite. Let $h : X \to \mathbb{R}^m$ be a G-map.

Define the metric d' on X by $d'(x,y) := \max\{d(x,y), \|h(x) - h(y)\|\}$ where d is the length metric on X.

With the metric d', G acts on X freely and properly by isometries. Give $G\backslash X$ the induced metric $\bar{d}$. Then $p : X \to G\backslash X$ is a local isometry.

We are given $\mu > 0$. We wish to triangulate X so that each simplex σ has diameter $< \mu$. Let $\alpha > 0$ be as in the definition of local isometry, and let $\delta := \min\{\mu, \alpha\}$. Cover $G\backslash X$ with $\mathcal{U} := \{N_\delta(x) | x \in G\backslash X\}$. There is a triangulation K of $G\backslash X$ finer than $\mathcal{U}$. Lift this to a triangulation $\tilde{K}$ of X; then for each simplex σ of $\tilde{K}$, $\text{diam}(\sigma) < \mu$.

Theorem 3.9 *If ρ is BC^{n-1} in the direction $e \in \partial_\infty \mathbb{R}^m$, then ρ is CC^{n-1} in all directions which are in an open $\frac{\pi}{2}$-neighborhood of e.*

PROOF: Let ρ be BC^{n-1} in the direction $e \in \partial_\infty \mathbb{R}^m$ and let e' be in the $\frac{\pi}{2}$-neighborhood of e with $\angle(e, e') = \beta < \frac{\pi}{2}$. Let γ' be a geodesic ray defining e', and let $\mu > 0$. Choose an n-dimensional, $(n-1)$-connected, free G-simplicial complex X with $G\backslash X$ finite, and choose a G-map $h : X \to \mathbb{R}^m$. Suppose X has the metric d' described above, and suppose X is subdivided so that $\text{diam}(\sigma) < \mu$ for each simplex σ. Let $\varepsilon > 0$, let $\ell := n^2$, and let $\nu(\mu)$ be as in Lemma 3.6 (without loss of generality, assume $\nu(\mu) \le \mu$ for every $\mu > 0$).

Since ρ is BC^{n-1} in the direction e, there exists $r_0 \ge 0$ and a function $\lambda : [r_0, \infty) \to [0, \infty)$ such that $r - \lambda(r) \to \infty$ as $r \to \infty$, and for each $-1 \le p \le n-1$ and for every geodesic ray γ defining e, every map $f : S^p \to \mathring{Y}_{\gamma, r}$ extends to a map $\hat{f} : B^{p+1} \to \mathring{Y}_{\gamma, r-\lambda}$. Since $r - \lambda(r) \to$

∞, there exists $R_{\ell-1} > r_0$ such that $R_{\ell-1} - \lambda(R_{\ell-1}) > \tan(\beta)$. Let $R_\ell := R_{\ell-1} - \lambda(R_{\ell-1})$. Similarly, we can find $R_{\ell-2}, \ldots, R_0$ such that for each $0 \leq i \leq \ell - 2$, $R_i - \lambda(r_i) \geq r_{i+1}$. Thus, $R_0 \geq R_1 \geq \ldots \geq R_\ell$. Let $\lambda' = 1 + R_0 - R_\ell + 2\mu + \varepsilon$.

Let $H_{\gamma',s}$ be a half-space, $-1 \leq p \leq n-1$, and $f : S^p \to X_{\gamma',s}$ be a map. Since $H_{\gamma',s}$ is contractible, we can extend $h \circ f$ to a map $F : B^{p+1} \to H_{\gamma',s}$. For each $a \in F(B^{p+1})$, define γ_a to be the geodesic ray defining e with $\gamma_a(R_0 + \varepsilon) = a$. For each $0 \leq j \leq \ell$, define $\mathcal{S}_j := \{\overset{\circ}{Y}_{R_j,\gamma_a} \,|a \in F(B^{p+1})\}$. For each $0 \leq j \leq \ell$, define $\varphi_j : B^{p+1} \to \mathcal{S}_j$ by $\varphi_j(t) := \overset{\circ}{Y}_{\gamma_a,R_j}$ where $F(t) = a$.

Each $\overset{\circ}{C}_{\gamma_a,R_\ell} \subseteq H_{\gamma',s-\lambda'+2\mu}$. To see this, let $y \in base(\overset{\circ}{C}_{\gamma_a,R_\ell})$ and let c be the center of $base(\overset{\circ}{C}_{\gamma_a,R_\ell})$. We know that $H_{\gamma',s} \cap \gamma_a([0,\infty)) = \gamma_a([t,\infty))$ for some $t \leq R_0 + \varepsilon$. Thus, $d(y, H_{\gamma',s}) \leq d(y, \gamma_a(t)) \leq d(y,c) + d(c, \gamma_a(t)) \leq 1 + t - R_\ell \leq \lambda' - 2\mu$, so $base(\overset{\circ}{C}_{\gamma_a,R_\ell}) \subseteq H_{\gamma',s-\lambda'+2\mu}$. We also know that $R_\ell > \tan(\beta)$, so $\arctan\left(\frac{1}{R_\ell}\right) < \arctan\left(\frac{1}{\tan(\beta)}\right) = \frac{\pi}{2} - \beta$. Therefore, the angle of $\overset{\circ}{C}_{\gamma_a,R_\ell}$ is less than $\frac{\pi}{2} - \beta$, so this fact together with $base(\overset{\circ}{C}_{\gamma_a,R_\ell}) \subseteq H_{\gamma',s-\lambda'+2\mu}$ implies that $\overset{\circ}{C}_{\gamma_a,R_\ell} \subseteq H_{\gamma',s-\lambda'+2\mu}$. Thus, $\overset{\circ}{Y}_{\gamma_a,R_\ell} \subseteq X_{\gamma',s-\lambda'+2\mu}$. Let $\lambda'(s) = \lambda'$. Since λ' was chosen independently from s, we have $s - \lambda'(s) \to \infty$ as $s \to \infty$.

We now check the hypotheses of Theorem 3.2 for the simplicial complex X, the maps $\varphi_i : B^{p+1} \to \mathcal{S}_i$, the set $\mathcal{S} := \{(\overset{\circ}{Y}_{\gamma_a,R_0}, \ldots, \overset{\circ}{Y}_{\gamma_a,R_\ell}) \in (\mathcal{S}_0, \ldots, \mathcal{S}_\ell) | a \in F(B^{p+1})\}$, and each $(\overset{\circ}{Y}_{\gamma_a,R_0}, \ldots, \overset{\circ}{Y}_{\gamma_a,R_\ell}) \in \mathcal{S}$.

The simplicial complex X is uniformly LC^{n-1} (with respect to d') since $G\backslash X$ is finite and $p : X \to G\backslash X$ is a local isometry. Therefore, X satisfies condition (1). To see that each φ_i is l.s.c., let V be open in X, and suppose $\varphi_i(t) \cap V \neq \emptyset$ for some $t \in B^{p+1}$. Since $h(V \cap \varphi_i(t)) \subset \overset{\circ}{Y}_{\gamma_a,R_i}$ and $\overset{\circ}{Y}_{\gamma_a,R_i}$ is open, for every $b \in h(V \cap \varphi_i(t))$, the distance from b to the boundary of $\overset{\circ}{Y}_{\gamma_a,R_i}$ is positive. Pick $b \in h(V \cap \varphi_i(t))$ and let $D > 0$ be the distance from b to the boundary of $\overset{\circ}{Y}_{\gamma_a,R_i}$. Let $U := F^{-1}(N_{D/2}(F(t)))$ and let $t' \in U$. Then $\varphi_i(t') = \overset{\circ}{Y}_{\gamma'_a,R_i}$ where $a' = F(t')$. Since $\overset{\circ}{Y}_{\gamma'_a,R_i}$ is a translation of $\overset{\circ}{Y}_{\gamma_a,R_i}$, we have $b \in \overset{\circ}{Y}_{\gamma'_a,R_i}$ so $V \cap \varphi_i(t') \neq \emptyset$, hence φ_i is a l.s.c. function. Thus, condition (2) is satisfied. Since G acts on X freely and properly by isometries, $\mathcal{S}$ is a uniformly equi-LC^{n-1}, and condition (3) is satisfied. Clearly Condition (4) is satisfied.

By Theorem 3.2, there exists a ν-selection g for φ_ℓ such that $g(t) \in N_\nu(f(t))$ for each $t \in S^p$. By Lemma 3.6, there is a homotopy $H : S^p \times I \to X$ such that $H(t,s) \in N_\mu(f(t))$ for each $t \in S^p$ and each $s \in I$. This homotopy and the ν-selection give an extension $\hat{f}$ of f such that for each $x \in \hat{f}(B^{p+1})$, $x \in N_\mu(X_{\gamma',s-\lambda'+2\mu})$. Therefore, for each $x \in \hat{f}(B^{p+1})$, we have $h(x) \in N_\mu(H_{\gamma',s-\lambda'+2\mu})$. Therefore, $h \circ \hat{f}(B^{p+1}) \subseteq H_{\gamma',s-\lambda'+\mu}$, so $\hat{f}(B^{p+1}) \subseteq h^{-1}(H_{\gamma',s-\lambda'+\mu})$. However, since each simplex of X has diameter $< \mu$, the largest subcomplex $X_{\gamma',s-\lambda'}$ of $h^{-1}(H_{\gamma',s-\lambda'})$ contains $N_\mu(h^{-1}(H_{\gamma',s-\lambda'+\mu}))$ Thus, $\hat{f}(B^{p+1}) \subseteq X_{\gamma',s-\lambda'}$. □

3.3 Sheaves of maps

In this section, we review sheaves of maps and some related ideas as discussed in [1, Chapters 4, 13, and 14]. Let X and Y be CW-complexes. A *sheaf (of maps)* $\mathcal{F} : X \rightsquigarrow Y$ is a set $\mathcal{F}$ of cellular maps $f : f(D) \to Y$ with domain $D(f)$, a finite subcomplex of X, satisfying the following axioms:

1. $\mathcal{F}$ contains the empty map.
2. If $f \in \mathcal{F}$ and if K is a finite subcomplex of $D(f)$, then $f|_K$ is also in $\mathcal{F}$.
3. If $f, f' \in \mathcal{F}$ and they agree on the intersection of their domains, then $f \cup f' : D(f) \cup d(f') \to Y$ is in $\mathcal{F}$.

If K is a subcomplex of X, then $\mathcal{F}|K$ denotes the sheaf consisting of all restrictions of maps in $\mathcal{F}$ to subcomplexes of K. Given a cellular map $\phi : X \to Y$. Denote by $\text{Res}(\phi)$ the set of all restrictions of ϕ to the finite subcomplexes of X. A *cross section of a sheaf* $\mathcal{F}$ is a cellular map $\phi : X \to Y$ whose sheaf $\text{Res}(\phi)$ is a subsheaf of $\mathcal{F}$. A sheaf $\mathcal{F}$ is *homotopically closed* if for each $f \in \mathcal{F}$ and each finite subcomplex $K \supseteq D(f)$, there exists $\hat{f} \in \mathcal{F}$ such that $D(\hat{f}) = K$ and $\hat{f}|_{D(f)} = f$.

Now suppose X and Y are G-CW complexes. Then G acts on the set of all cellular maps $\phi : K \to Y$ with K a subcomplex of X by $g\phi$ is the map with domain $D(g\phi) = gK$ and $(g\phi)(x) = g\phi(g^{-1}x)$ for each $x \in gK$. A *G-sheaf* is a sheaf that is closed under this action. A sheaf $\mathcal{F}$ is *locally finite* if for each finite subcomplex K of X, the restriction $\mathcal{F}|_K$ is a finite set of maps. A cellular map $\phi : X \to Y$ is *G-finitary* if ϕ is a cross section of a locally finite G-sheaf $\mathcal{F} : X \rightsquigarrow Y$.

Suppose G is acting on $\mathbb{R}^m$ by translations, and suppose X and $h : X \to \mathbb{R}^m$ are as in § 2.1. Given a map $f : D(f) \to X$ with $D(f)$ a subcomplex of X. Define $\alpha_f : D(f) \to [0, \infty)$ by $\alpha_f(x) := d(h(x), hf(x))$. Define the *norm* of f to be $\|f\| := \sup\{\alpha_f(x) | x \in D(f)\}$. The norm $\|f\|$ may be infinite if $D(f)$ is not finite. The map f is *bounded* if $\|f\| < \infty$. If ϕ is the empty map, then put $\|\phi\| = 0$. Let $\mathcal{F} : X \rightsquigarrow X$ is a locally finite sheaf, and let $D(\mathcal{F})$ denote the union of all domains $D(f)$, $f \in \mathcal{F}$. Define $\alpha_{\mathcal{F}} : D(\mathcal{F}) \to [0, \infty)$ by $\alpha_{\mathcal{F}}(x) := \sup\{\alpha_f(x) | f \in \mathcal{F}\}$, and define $\|\mathcal{F}\| := \sup\{\alpha_{\mathcal{F}}(x) | x \in D(\mathcal{F})\}$. Again, $\mathcal{F}$ is *bounded* if $\|\mathcal{F}\| < \infty$.

The *shift of f towards $e \in \partial_\infty \mathbb{R}^m$* is the function $sh_{f,e} : D(f) \to \mathbb{R}$ defined by $sh_{f,e}(x) := \beta_\gamma hf(x) - \beta_\gamma h(x)$ where γ is a geodesic ray defining e and β_γ is defined in § 2.1. The shift is independent of the choice of γ, and $|sh_{f,e}(x)| \leq \alpha_f(x)$. The *guaranteed shift towards e* is $gsh_e(f) := \inf\{sh_{f,e}(x) | x \in D(f)\}$. For each $g \in G$, we have $gsh_e(gf) = gsh_e(f)$. A cellular map $\phi : X \to X$ is a *contraction towards e* if $gsh_e(\phi) > 0$.

Suppose $\mathcal{F} : X \rightsquigarrow X$ is a locally finite homotopically closed G-sheaf. Given a cell σ of X, the *maximal guaranteed vertex shift on σ* is $\mu_e(\mathcal{F}|\sigma) := \max\{gsh_e(f) | f \in \mathcal{F}, D(f) = C(\sigma)^0\}$ where $C(\sigma)$ is the carrier[6] of σ. The *defect of $\mathcal{F}$ towards e on σ* is $d_e(\mathcal{F}|\sigma) :=$

[6]The *carrier* of a simplex σ is the smallest subcomplex of X containing σ

$\max\{\min\{gsh_e(f) - gsh_e(\hat{f}) \big| D(\hat{f}) = C(\sigma),\ \hat{f} \text{ extends } f\} \big| D(f) = C(\dot{\sigma})\}$. The *total defect of* $\mathcal{F}$ *towards* e *on* σ is $\delta_e(\mathcal{F}|\sigma) := \sum \max\{d_e(\mathcal{F}|\tau) | \tau \text{ is a } j\text{-cell of } C(\sigma); 1 \leq j \leq \dim(\sigma)\}$.

3.4 The "if" direction

In this section we prove the "if" direction of Theorem 3.1 which is the following:

Theorem 3.10 *If* ρ *is* CC^{n-1} *in all directions in an open* $\frac{\pi}{2}$*-neighborhood of* e*, then* ρ *is* BC^{n-1} *in the direction* e*.*

We need:

Lemma 3.11 *Let* $\varepsilon > 0$*, let* E *be a closed subset of* $\partial_\infty \mathbb{R}^m$*, let* $\mathcal{F}$ *be a locally finite homotopically closed* G*-sheaf, and let* X *be an* n*-dimensional,* $(n-1)$*-connected free* G*-simplicial complex with* $G \backslash X$ *finite. For each* $e \in E$*, there exists an open neighborhood* N *of* e *such that for every simplex* σ *of* X*, if* $\phi \in \mathcal{F}$ *with* $sh_{\phi,e}(x) \geq \varepsilon$ *for all* $x \in \sigma$*, then for each* $e' \in N$*,* $sh_{\phi,e'}(x) > \frac{\varepsilon}{3}$ *for all* $x \in \sigma$*.*

PROOF: Suppose $e \in E$ and $R := \|\mathcal{F}\| \left(1 + \frac{12\|\mathcal{F}\|}{\varepsilon}\right) + \varepsilon$. Let $N := \{e' \in \partial_\infty \mathbb{R}^m | \angle(e, e') < 2 \arcsin\left(\frac{\varepsilon}{6R}\right)\}$. Let σ be a simplex of X, and suppose $\phi \in \mathcal{F}$ with $sh_{\phi,e}(x) \geq \varepsilon$ for all $x \in \sigma$. Let $e' \in N$, and suppose γ and γ' are geodesic rays defining e and e' respectively. By Euclidean geometry, $d(\gamma(R), \gamma'(R)) = 2R \sin\left(\frac{\angle(e,e')}{2}\right)$. Applying [1, Lemma 13.5] (with $\frac{\varepsilon}{6}$ replacing ε and $r = \|\phi\|$), we get that for each $x \in \sigma$ and each $p \in B_{\|\phi\|}(h(x))$, $|\beta_\gamma(p) - \beta_{\gamma'}(p)| < \frac{\varepsilon}{3} + d(\gamma(R), \gamma'(R))$. Thus, $|sh_{\phi,e}(x) - sh_{\phi,e'}(x)| = |\beta_\gamma \circ h \circ \phi(x) - \beta_{\gamma'} \circ h \circ \phi(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3R} R = \frac{2\varepsilon}{3}$. Therefore, $sh_{\phi,e'}(x) > \frac{\varepsilon}{3}$. □

PROOF OF THEOREM 3.10: Let $e \in \partial_\infty \mathbb{R}^m$, let X be an n-dimensional, $(n-1)$-connected free G-simplicial complex with $G \backslash X$ finite, and let $h : X \to \mathbb{R}^m$ be a G-map. Suppose ρ is CC^{n-1} in all directions in an open $\frac{\pi}{2}$-neighborhood of e. Let $r > 0$ and γ be a geodesic ray defining e. Define $\pi : (\mathbb{R}^m - C_{\gamma,r}) \to \partial_\infty \mathbb{R}^m$ as follows: for each $a \in \mathbb{R}^m - C_{\gamma,r}$, there is a unique closest point $b \in C_{\gamma,r}$. Let γ_a be the unique geodesic ray with $\gamma_a(0) = a$ and $\gamma_a(t) = b$ for some $t > 0$. Then $\pi(a) := \gamma_a(\infty)$. The set $E = \pi(\mathbb{R}^m - C_{\gamma,r})$ is a closed subset of $\partial_\infty \mathbb{R}^m$ which is contained in the open $\frac{\pi}{2}$-neighborhood of e (see Figure 3).

By [1, Theorem 14.5], there is a locally finite homotopically closed G-sheaf $\mathcal{F} : X \rightsquigarrow X$ and $\varepsilon > 0$ such that $\mu_{e'}(\mathcal{F}|\sigma) - \delta_{e'}(\mathcal{F}|\sigma) \geq \varepsilon$ for each simplex σ of X and each $e' \in E$. Let $\hat{\mathcal{F}}$ be the G-sheaf generated by $\mathcal{F}$ and id: $X \to X$. Therefore, $\mu_{e'}(\hat{\mathcal{F}}|\sigma) - \delta_{e'}(\hat{\mathcal{F}}|\sigma) \geq \varepsilon$ for each simplex σ of X and each $e' \in E$. Use the neighborhoods N in Lemma 3.11 to cover E. Therefore, there is a finite set $\{e_1, \ldots, e_k\} \subseteq E$ such that for each $e' \in E$, there exists $1 \leq j \leq k$ such that for every simplex σ of X, if $\phi \in \hat{\mathcal{F}}$ with $sh_{\phi,e_j}(x) \geq \varepsilon$ for all $x \in \sigma$, then $sh_{\phi,e'}(x) > \frac{\varepsilon}{3}$ for all $x \in \sigma$.

Since $G \backslash X$ is finite, there are a finite number of orbits of vertices. Let $u_1, \ldots, u_m$ denote the representatives of these orbits. Let $u \in \{u_1, \ldots, u_m\}$, and let $e_j \in \{e_1, \ldots, e_k\}$. We will

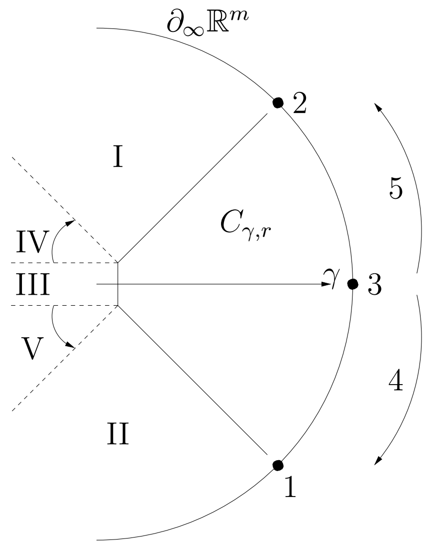

Figure 3: The map $\pi : (\mathbb{R}^m - C_{\gamma,r}) \to \partial_\infty\mathbb{R}^m$ maps Region I in $\mathbb{R}^m$ to the point 1 in $\partial_\infty\mathbb{R}^m$, it maps Region II to the point 2, and it maps Region III to the point 3 (which is e). It maps Region IV continuously (following the arrow) onto Region 4 in $\partial_\infty\mathbb{R}^m$ (following the arrow), and it maps Region V continuously onto Region 5.

now construct a map $\phi : X \to X$ with $sh_{\phi,e_j}(x) \geq \mu_{e_j}(\hat{\mathcal{F}}|\sigma) - \delta_{e_j}(\hat{\mathcal{F}}|\sigma)$ for each simplex σ of $\mathrm{St}(u)$ and each $x \in \sigma$. For each vertex $v \in (\mathrm{St}(u))^0$, choose $f_v \in \hat{\mathcal{F}}$ with $v = D(f_v)$ and $sh_{f_v,e_j}(v) = gsh_{e_j}(f_v) = \mu_{e_j}(\hat{\mathcal{F}}|\{v\})$. Define $\phi^{(0)} : X^0 \to X^0$ by $\phi_{(0)}(v) = f_v(v)$ if $v \in (\mathrm{St}(u))^0$ and $\phi_{(0)}(v) = v$ otherwise. Therefore, $gsh_{e_j}(\phi_{(0)}|K^0) = \mu_{e_j}(\hat{\mathcal{F}}|K^0)$ for all finite subcomplexes K^0 of $(\mathrm{St}(u))^0$.

Assume for $k \geq 1$ that $\phi_{(k-1)} : X^{k-1} \to X^{k-1}$ has been constructed. Let σ be a k-simplex of X. If $\phi_{(k-1)}|_{\overset{\bullet}{\sigma}} = \mathrm{id}$, then define $\phi_{(k)}|_\sigma = \mathrm{id}$. Otherwise, extend $\phi_{(k-1)}$ to σ by $\tilde{f} : \sigma \to X$ with $gsh_{e_j}(\tilde{f}) \geq gsh_{e_j}(\phi_{(k-1)}|\overset{\bullet}{\sigma}) - d_{e_j}(\hat{\mathcal{F}}|\sigma)$. Thus, $\phi_{(k)} : X^K \to X^k$ has the property that for each finite subcomplex $K \subseteq (\mathrm{St}(u))^k$, $gsh_{e_j}(\phi_{(k)}|K) \geq gsh_{e_j}(\phi_{(k)}|K \cap X^{k-1}) - \max\{d_{e_j}(\hat{\mathcal{F}}|\sigma)|\sigma$ is a k-simplex of $K\}$. By induction on $k \geq 1$, $gsh_{e_j}(\phi_{(k)}|K) \geq \mu_{e_j}(\hat{\mathcal{F}}|K^0) - \sum \max\{d_{e_j}(\hat{\mathcal{F}}|\sigma)|\sigma$ is a j-simplex of $K; 1 \leq j \leq k\}$. Therefore, there is a map $\phi : X \to X$ such that $sh_{\phi,e_j}(x) \geq \mu_{e_j}(\hat{\mathcal{F}}|\sigma) - \delta_{e_j}(\hat{\mathcal{F}}|\sigma)$ for each simplex σ of $\mathrm{St}(u)$ and each $x \in \sigma$.

Given $u \in \{u_1, \ldots, u_m\}$ and $e' \in E$, let γ' be the geodesic ray defining e' with $\gamma'(0) = u$. Define $(\mathrm{St}(u))^+_{\gamma'} := \mathrm{St}(u) \cap \beta^{-1}_{\gamma'}([0,\infty))$. Given $s, t \in \mathbb{R}$ with $s < t$, let $H_{\gamma',[s,t]} := \beta^{-1}_{\gamma'}([s,t])$, and let $\overset{\circ}{X}_{\gamma',[s,t]}$ denote the largest subcomplex of X contained in $h^{-1}(H_{\gamma',[s,t]})$. Since there are a finite number of orbits of vertices and a finite set $\{e_1, \ldots, e_k\}$, there is a finite number of maps $\{\phi_1, \ldots, \phi_p\}$ constructed as above. Thus, there exists $\alpha < \frac{\pi}{2}$ and there exists $\delta > \varepsilon$ such that for each $e' \in E$ and each $u_i \in \{u_1, \ldots, u_m\}$, there exists $\phi_t : X \to X$ such that for each simplex σ in $(\mathrm{St}(u))^+_{\gamma'}$, $\phi_t(\sigma) \subseteq \overset{\circ}{X}_{\gamma',[\varepsilon/3,\delta]} \cap D(\alpha,\gamma')$ where $D(\alpha,\gamma') := h^{-1}(Cone_\alpha(\gamma')$.

Let $\eta = \frac{\delta}{2}\sec^2(\alpha)$. There exists $\lambda \geq 0$ such that $N_\eta(C_{\gamma,r}) \subseteq C_{\gamma,r-\lambda}$. Let $f : S^p \to Y_{\gamma,r}$ be a map. Since ρ is CC^{n-1} in the direction e, there is an extension $\tilde{f} : B^{p+1} \to X_{\gamma,r}$. Let K be the smallest subcomplex of X containing $\tilde{f}(B^{p+1})$. Let $v \in K^0$ be a vertex with maximal distance from $C_{\gamma,r}$, let $gu_i = v$ for some $g \in G$ and $u_i \in \{u_1, \ldots, u_m\}$, and let $a \in C_{\gamma,r}$ be the closest point in $C_{\gamma,r}$ to v. Apply Lemma 3.12, below, to get $(g\phi_t g^{-1}) \circ \tilde{f}$. Proceed by induction to obtain $\hat{f} : B^{p+1} \to Y_{\gamma,r-\lambda}$ which is the desired extension. □

It remains to prove:

Lemma 3.12 *Let $a \in \mathbb{R}^m$, and let $v \in X^0$ with $d(h(v), a) > \frac{\delta}{2}\sec^2(\alpha)$. Let γ' be the geodesic ray starting at $h(v)$ that passes through a, and let $e' = \gamma'(\infty)$. Let σ be a simplex of $(\mathrm{St}(v))^+_{\gamma'}$, let $\phi_t \in \{\phi_1, \ldots, \phi_p\}$ such that $\phi_t(\sigma) \subseteq \mathring{X}_{\gamma',[\varepsilon/3,\delta]} \cap D(\alpha, \gamma')$, and let $gu_i = v$ for some $g \in G$ and $u_i \in \{u_1, \ldots, u_m\}$. Then every $x \in g\phi_t(g^{-1}\sigma)$ satisfies $d(h(x), a) < d(h(v), a)$.*

PROOF: Let $d(a, h(v)) = \ell > \frac{\delta}{2}\sec^2(\alpha)$, so $-2\delta\ell < -2\delta^2\sec^2(\alpha)$. Let $y = \delta\tan(\alpha)$, so

$$\begin{aligned} R &= \sqrt{(\ell-\delta)^2 + y^2} \\ &= \sqrt{\ell^2 - 2\delta\ell + \delta^2 + \delta^2\tan^2(\alpha)} \\ &= \sqrt{\ell^2 - 2\delta\ell + \delta^2\sec^2(\alpha)} \\ &< \sqrt{\ell^2 - \delta^2\sec^2(\alpha) + \delta^2\sec^2(\alpha)} \\ &= \ell. \end{aligned}$$

Therefore, $b \in N_\ell(g^{-1}a)$ and $h(u_i) \in B_\ell(g^{-1}a)$ (see Figure 4), so the line segment from b to $h(u_i)$ is contained in $B_\ell(g^1 a)$. It follows that $h(\mathring{X}_{g^{-1}\gamma',[\varepsilon/3,\delta]} \cap D(\alpha, g^1\gamma')) \subseteq N_\ell(g^{-1}a)$. Since each $g \in G$ is a translation, we have $h(\mathring{X}_{\gamma',[\varepsilon/3,\delta]} \cap D(\alpha, \gamma')) \subseteq N_\ell(a)$. Thus, if $x \in g\phi_t(g^{-1}\sigma)$, then $d(h(x), a) < d(h(v), a)$. □

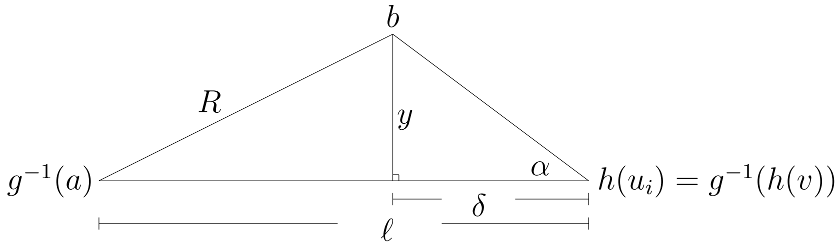

Figure 4: $h(\mathring{X}_{g^{-1}\gamma',[\varepsilon/3,\delta]} \cap D(\alpha, g^{-1}\gamma'))$ is contained in the triangle on the right. Thus, if $b \in N_\ell(g^{-1}a)$, then by convexity, $h(\mathring{X}_{g^{-1}\gamma',[\varepsilon/3,\delta]} \cap D(\alpha, g^{-1}\gamma')) \subseteq N_\ell(g^{-1}a)$.

Corollary 3.13 *The set $\Omega^n(\rho)$ is a closed subset of $\partial_\infty\mathbb{R}^m$.*

PROOF: Suppose $e \in (\Omega^n(\rho))^c$. Then by Theorem 3.1, there exists e' in the open $\frac{\pi}{2}$-neighborhood of e such that $e' \in (\Sigma^n(\rho))^c$. This implies that ρ is not BC^{n-1} in any direction in an open $\frac{\pi}{2}$-neighborhood of e', so this neighborhood is contained in $(\Omega^n(\rho))^c$. Since e is in the $\frac{\pi}{2}$-neighborhood of e', we have found an open neighborhood of e that is contained in $(\Omega^n(\rho))^c$. □

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