The Σ^1-Invariants of B_3

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Abstract

We calculate the Bieri-Neumann-Strebel invariant, Σ^1, of the braid group on three strands, $B_3 \cong \langle a, b | aba = bab \rangle$. We provide a method of constructing the Cayley graph of B_3, and show that the group of homomorphisms from B_3 to the additive reals is isomorphic to $\mathbb{R}$. We may map the Cayley graph of B_3 into $\mathbb{R}$ by defining an appropriate height function, and then compute $\Sigma^1(B_3)$ by showing that the Cayley graph of B_3 is path-connected at $\pm\infty$.

Keywords: Σ-invariants, braid groups

1 The Braid Group on Three Strands

Our group of interest is B_3, the braid group on three strands. The group operation of B_3 is concatenation. By stacking two braids σ_1 and σ_2 on top of one another, we obtain yet another braid, σ_3, as in Figure 1, and thus the group is closed.

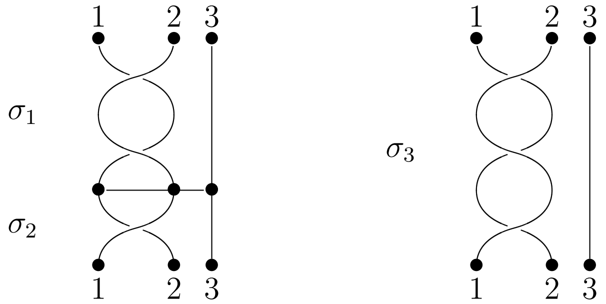

Figure 1: Concatenation of two braids to obtain a new braid.

The order in which the strands pass over and under each other is important as this gives rise to the notion of a braid's inverse. Since the braid diagrams are isotropic, one can imagine wiggling the strands in Figure 2 apart to obtain the identity element as shown.

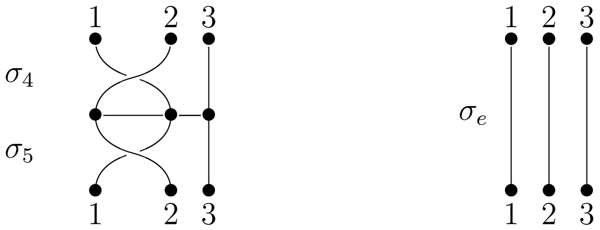

Figure 2: Here we have the product $\sigma_4\sigma_5$ where $\sigma_5 = \sigma_4^{-1}$. Note that $\sigma_4\sigma_5 = \sigma_e$ where σ_e represents the identity braid.

Associativity of braids is trivial and thus we need not verify this property. It is helpful to think of braids as representing permutations. Consider the diagram below.

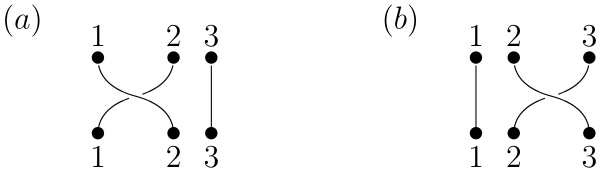

Figure 3: Generators of B_3.

If we label the endpoints of these strands, we can follow the path of the strands to see where they end up. In (a), $1 \mapsto 2$, $2 \mapsto 1$, and $3 \mapsto 3$. In (b), $1 \mapsto 1$, $2 \mapsto 3$, and $3 \mapsto 2$. Given any braid σ, it is known that the entire braid can be represented as a series of concatenations of a's, b's, and their formal inverses. Thus we say that B_3 is generated by a and b. It is also worth noting that a braid aba is equivalent to a braid bab, and thus we have the relation $aba = bab$ as shown in Figure 4. Therefore we say that B_3 has presentation $B_3 \cong \langle a, b | aba = bab \rangle$.

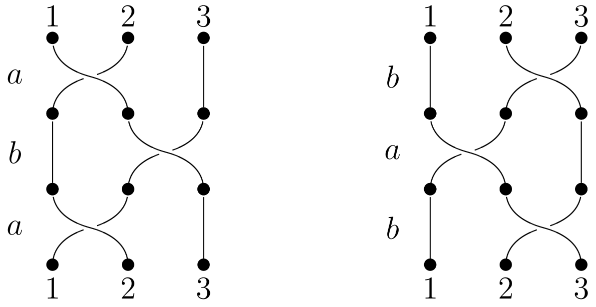

Figure 4: Two equivalent braids, aba and bab, which form the only relation in B_3.

2 How to Compute $\Sigma^1(G)$

Before we compute the Bieri-Neumann-Strebel geometric invariant, we must define the invariant as in [1], as well as familiarize ourselves with the notion of a boundary at infinity, a Cayley graph, and a height function. We present these definitions here.

2.1 The boundary at infinity

Consider two geodesic rays

$$\gamma : [0, \infty) \to \mathbb{R}^n,$$
$$\gamma' : [0, \infty) \to \mathbb{R}^n.$$

We say $\gamma \sim \gamma'$ if and only if $\forall t \in [0, \infty), d(\gamma(t), \gamma'(t)) = d(\gamma(0), \gamma'(0))$. Define the set $\mathcal{R}$ as

$$\mathcal{R} = \{\textit{all geodesic rays in } \mathbb{R}^n\}.$$

Then the relation $\sim$ is an equivalence relation on $\mathcal{R}$. We define $\partial_\infty \mathbb{R}^n = \mathcal{R}/\sim$ as the boundary of $\mathbb{R}^n$ at infinity. Thus we define a point $e \in \partial_\infty \mathbb{R}^n$ as $e = [\gamma]$.

2.2 Cayley graph of a group

Let G be a group with presentation $G = \langle X | R \rangle$ where X is the set of generators of the group and R is the set of relations in the group. A Cayley graph of the group G is composed of a set of vertices V and a set of edges E which connect the vertices. In a Cayley graph, $V = G$. Let $g_1, g_2 \in G$. We define $E = \{g_1, g_2 \mid \exists$ a generator $x \in X$ st $g_1 x = g_2\}$.

2.3 Height function

Let Γ be the Cayley graph of an infinite group G. Consider the real vector space $Hom(G, \mathbb{R}) \cong \mathbb{R}^n$ which is defined as the set of homomorphisms from an infinite group G into $\mathbb{R}$, the additive reals. We wish to define a function $h : \Gamma \to Hom(G, \mathbb{R})$ which we call a height function. Once we have defined the appropriate height function we may map Γ into $\mathbb{R}^n$. Let $s \in \mathbb{R}^n$. The point $\gamma(s)$ is the point on γ that corresponds to s. If we take a line ℓ orthogonal to γ through the point $\gamma(s)$, we have constructed a half-space, denoted $H_s(\gamma)$, of $\mathbb{R}^n$. We also define $\Gamma_s(\gamma)$ to be the smallest subgraph of Γ containing $h^{-1}(H_s(\gamma))$.

2.4 The Bieri-Neumann-Strebel geometric invariant Σ^1

Definition: A point $e \in \partial_\infty \mathbb{R}^n$ is CC^0 (*controlled 0-connected*) if and only if $\forall s \geq 0,\ \Gamma_s(\gamma)$ is 0-connected (path-connected).

For a group G, the Bieri-Neumann-Strebel geometric invariant is $\Sigma^1(G) = \{e \in \partial_\infty \mathbb{R}^n | e \textit{ is } CC^0\}$.

2.5 $Hom(B_3, \mathbb{R})$, the boundary at infinity, and our height function

For a homomorphism $\phi \in Hom(B_3, \mathbb{R})$, and for the generators $a, b \in B_3$, $\exists r_a, r_b \in \mathbb{R}$ st $\phi(a) = r_a$ and $\phi(b) = r_b$. According to the relation $aba = bab$ in B_3, we have $\phi(aba) = \phi(bab)$. Since ϕ is a homomorphism from B_3 to the additive reals, we have that

$$\phi(a) + \phi(b) + \phi(a) = \phi(b) + \phi(a) + \phi(b).$$

Thus,

$$r_a + r_b + r_a = r_b + r_a + r_b$$

and therefore

$$r_a = r_b.$$

We conclude then that the generators of B_3 get mapped to the same real number. For convenience, we will say that $r_a = r_b = 1$. Thus it follows that $Hom(B_3, \mathbb{R}) \cong \mathbb{R}$. Therefore $\partial_\infty \mathbb{R} = \{\pm\infty\}$.

We define our height function $h : \Gamma_{B_3} \to \mathbb{R}$ as follows:
(1) For vertices, $h(w) = \exp(w)$ where $\exp(w)$ is the exponential sum of the a's and b's representing the word w and
(2) extend linearly on edges.

3 The Cayley Graph of B_3

Here we construct the Cayley graph of B_3 as in [2]. Consider the subgroup of B_3, $\langle a^2, b^2\rangle$, which is isomorphic to F_2. The construction of this graph is motivated by the construction of $F_2 \times \mathbb{Z}$, which is isomorphic to the pure braid group P_3, where P_3 is defined as the group of all braids that map to the identity. Let Γ_{B_3} denote the Cayley graph of B_3. $\forall k \in \mathbb{Z}$, define the vertex set of Γ_{B_3}, here denoted as $V(\Gamma_{B_3})$, to be $V(\Gamma_{B_3}) = \bigsqcup\limits_{k\in\mathbb{Z}}(V(\langle a^2, b^2\rangle)) \times \{k\}$. Define the edge set of B_3 as $E(\Gamma_{B_3}) = E(\langle a^2, b^2\rangle) \times \{k\}$. Let $v, w \in V(\Gamma_{B_3})$. Define an edge $e \in E(\Gamma_{B_3})$ where e is labelled by s to be $e = [v, w, s]$.

A vertex $v \in \langle a^2, b^2\rangle$ is called an a-vertex (respectively, b-vertex) if v has exactly two incident positive edges $\langle a^2, b^2\rangle$, both labelled by a (respectively, b). Define the map $\lambda : S^* \to S^*$, where S^* is the set of all possible words composed of a's and b's and their formal inverses, to be the map induced by $\lambda(a) = b, \lambda(b) = a, \lambda(a^{-1}) = b^{-1}, \lambda(b^{-1}) = a^{-1}$. It follows that $\lambda^2(w) = w$. Consider any $v \in V(\langle a^2, b^2\rangle)$ with a path w from 1 to v in $\langle a^2, b^2\rangle$ for some $w \in S^*$. If v is an a-vertex (respectively, b-vertex), then let v' be the vertex represented by $\lambda(w)b^{-1}a^{-1}$ (respectively, $\lambda(w)a^{-1}b^{-1}$) in $\langle a^2, b^2\rangle$.
Definition(The graph Γ_{B_3}). Define the graph Γ_{B_3} as follows:

- $V(\Gamma_{B_3}) = \bigsqcup\limits_{k\in\mathbb{Z}}(V(\langle a^2, b^2\rangle)) \times \{k\}$
- The set of positively labelled edges of Γ_{B_3} is $(\bigcup_{k\in\mathbb{Z}}(E(\langle a^2, b^2\rangle)) \times \{k\}) \cup (\bigcup_{(v,k)\in V(\Gamma_{B_3})}[(v,k),(v',k+1),a] | v \textit{ is a } b\textit{-vertex}) \cup (\bigcup_{(v,k)\in V(\Gamma_{B_3})}[(v,k),(v',k+1),b] | v \textit{ is an } a\textit{-vertex})$.

4 The Main Result

Theorem: $\Sigma^1(B_3) = \{\pm\infty\}$
<u>Proof</u>:
In this proof we consider only the $+\infty$ direction. A similar argument for the $-\infty$ direction can be made.

$(+\infty)$: Let γ^+ be a geodesic ray on the interval $[0, \infty)$. Let $\Gamma^+_{B_3} = h^{-1}(\gamma^+)$ where $\Gamma^+_{B_3}$ is the full subgraph of the Cayley graph of B_3 with $\exp(v) \geq 0\ \forall v \in V(\Gamma^+_{B_3})$. Thus $\Gamma^+_{B_3}$ is the pre-image of γ^+.

Let $x, y \in V(\Gamma^+_{B_3})$. Let ℓ_v denote the level of $\Gamma^+_{B_3}$ that contains the subtree of F_2 spanned by $\langle a^2, b^2\rangle$ and in which a vertex $v \in \Gamma^+_{B_3}$ is contained. If $\ell_x = \ell_y$, then x and y lie on the same level.

If $\ell_x = \ell_y$ and there exists a geodesic path from x to y in $h^{-1}(\gamma^+)$, then the proof is complete. Suppose $\ell_x \neq \ell_y$.

Let ℓ_0 be the level of Γ_{B_3} that contains the vertex that corresponds to the identity element. Since each level in Γ_{B_3} is a tree, then $\forall w \in \ell_0, w = a^{k_1}b^{k_2}\ldots a^{k_{n-1}}b^{k_n}(aba)^0$ with $k_i > 1$ for $1 < i < n$ and $k_1, k_n \geq 0$ since no *aba* (equivalently, *bab*) relations can occur within the tree. Since every $g \in B_3$ is of the form $g = a^{k_1}b^{k_2}\ldots a^{k_{n-1}}b^{k_n}(aba)^j$ with $j \in \mathbb{Z}, k_i > 1$ for $1 < i < n$, and $k_1, k_n \geq 0$, then $\forall j \in \mathbb{Z}$, ℓ_j is the level that is j levels from the base level ℓ_0 (i.e. $\ell_j = \ell_{0+j}$). Since incrementing (respectively, decrementing) j corresponds to moving up (respectively, down) levels of Γ_{B_3} and this movement is integral in determining path-connectedness, then we give this movement a name.

<u>Definition</u>: Let v, v' be vertices with $v \in \ell_j$ and $v' \in \ell_{j+i}$ for some $i, j \in \mathbb{Z}$. We say v is <u>*i-relocated*</u> if $v' = v(aba)^i$.

Thus we have that $h(w(aba)^i) = h(w) + 3i > 0$ for some $i \in \mathbb{Z}^+$, and thus $\ell_y = \ell_{x+i}$ when $x < y$. We can see that a point x on ℓ_x can be relocated to ℓ_y by moving along a path composed of multiples of *aba*. Once x is relocated, denoted x', one must connect x' and y by a positive path $\rho_+ \in E(\Gamma^+_{B_3})$. Assume x' and y cannot be connected by a geodesic path in $\ell_{x'} = \ell_y$.

Let ρ_+ be a sequence of vertices from x' to y. We have that $h(\rho_+) \subseteq^{compact} \mathbb{R}$ because $\rho_+ \subseteq^{compact} \Gamma^+_{B_3}$. Thus $h(\rho_+)$ is closed and bounded because $\exists r \in \mathbb{R}$ *st* $h(\rho_+) = [r, s]$ where $r \leq s$. Since $r < 0, r \notin h^{-1}(\gamma^+)$. We must relocate x' and y to vertices x'' and y' so that $\ell_{x''} = \ell_{y'}$ and so that there exists a geodesic path ρ'_+ from x'' to y' which lies completely within $h^{-1}(\gamma^+)$. In order to do so, we consider the $\min\{z_{\rho_+}\} = z_0 \in \mathbb{Z}^-$ for all $z_{\rho_+} \in \rho_+$. We wish to i-relocate the vertex corresponding to z_0 so that $z_0 + 3i \geq 0$ for some $i \in \mathbb{Z}^+$. In doing so, we must i-relocate all z_{ρ_+} which will force the exponential sum of all vertices in the path to become non-negative. We must choose an appropriate value of i in order to satisfy these conditions, and additionally we must ensure that $i = 2j$ for some $j \in \mathbb{Z}^+$ so that $\lambda^i(z_0) = z_0$. Thus $\exists \rho'_+$ from x'' to y' on $\ell_{x''} = \ell_{y'}$, and the entire path consists of the path relocating x to x' to x'', the geodesic path between x'' to y', and the backward relocation of y' to y.$\blacksquare$

References

[1] R. Bieri, W. Neumann, and R. Strebel, *A Geometric Invariant of Discrete Groups*, Invent. Math. **90**, 1987.

[2] L. Sabalka, *Geodesics in the Braid Group on Three Strands*, Contemporary Mathematics, Volume **360**, 2004.

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