TWO GENERALIZATIONS OF BIASED GRAPHS: CIRCUIT SIGNATURES AND MODULAR TRIPLES OF MATROIDS, AND BIASED EXPANSIONS OF BIASED GRAPHS BY

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ABSTRACT

Label each edge of a graph with a group element. Call the labels *gains*, and call the graph with this labeling a *gain graph*. This dissertation contains generalizations of two concepts in Zaslavsky's theory of gain graphs: *lift matroids* and *group expansions*.

Dowling and Kelly showed how to use a linear class of circuits of a matroid M, whose members we call *balanced circuits*, to construct an elementary lift of M. In the case that M comes from a graph, Zaslavsky defined balanced circuits in terms of gains. I use gains on an arbitrary matroid to define balanced circuits and show that the lift construction can be carried out precisely when M is ternary. Instrumental to the proof are the concept of a modular triple of circuits and a theorem that uses modular triples to characterize four types of circuit signatures, three of which are known (weak orientations, orientations, and ternary signatures) and one of which is new (lifting signatures).

A group expansion of an ordinary graph is an example of a gain graph. To construct one, replace each edge of a graph by several edges, one bearing as gain each possible value in a group. We introduce the concept of a *group expansion of a gain graph*. Then we find a formula that relates the chromatic polynomials of a gain graph and its group expansions. Our main result is a generalization of this formula, in which group expansions of gain graphs are generalized to *biased expansions of biased graphs*. The inspiration for our definitions and theorems is results of Zaslavsky and of Ehrenborg and Readdy. Zaslavsky found our formula for group expansions of ordinary graphs. Ehrenborg and Readdy found a hyperplane analogue of our formula when gain graphs with real positive gains are expanded by a finite group of roots of unity.

Recently, Zaslavsky answered the question: When is a biased expansion of an ordinary graph actually a group expansion? We ask the same question about biased

expansions of biased graphs. We cannot answer this question, but we do provide some partial results.

To Mom and Dad

Acknowledgments

I thank my advisor, Tom Zaslavsky, for rejuvenating my love of math at a time when I was discouraged. Also, thank you for encouraging me to accomplish more than I thought I was capable of. I am a more well-rounded and fruitful mathematician because of your efforts.

To my husband Nic, thank you for your support. Earning my Ph.D. has been as much a psychological challenge as a mathematical one. I needed constant support in order to persevere, and that is what you provided.

Contents

Chapter 0

Introduction

0.1 About This Dissertation

A *biased graph* is a graph with some additional structure. An important example of a biased graph is a *gain graph*. Zaslavsky is developing a theory of gain and biased graphs in a series of papers entitled "Biased Graphs". In this dissertation, I generalize two concepts that are introduced in that series: lift matroids of gain graphs and biased expansions of ordinary graphs.

This dissertation is intended to be self-contained. I assume a rudimentary understanding of graph theory and matroid theory. The basics of biased graph theory are explained in Section 0.2.

Chapters 1 and 2 are completed projects. The lift matroid generalization appears in Chapter 1, and Chapter 2 contains the biased expansion generalization. The research project in Chapter 3, which is also about biased expansions, is far from completion. I look forward to continuing this research once my graduate school days end.

0.2 Graphs, Biased Graphs, and Their Matroids

Here we extract definitions from the "Biased Graphs" series that are necessary for this dissertation.

Let $\Gamma = (V, E)$ be a graph. In this dissertation, all vertex and edge sets are finite. Edges are *links* (two distinct endpoints) or *loops* (two coincident endpoints). The *order* of Γ is $|V|$. A *walk* is a sequence of vertices and edges,

$$P = (v_0, e_1, v_1, \ldots, e_l, v_l),$$

where $v_i \in V$, $e_i \in E$, and e_i is incident to v_{i-1} and v_i. A walk is a *closed path* if $l > 0$ and it has no repeated vertices or edges except that $v_l = v_0$. It is an *open path* if no vertices are repeated.

A *circle* is the edge set of a closed path. A *theta graph* is the union of three internally disjoint open paths with the same two endpoints. A *tight bracelet* is the union of two circles that intersect at precisely one vertex. A *loose bracelet* is the union of two vertex-disjoint circles. A *loose handcuff* is the union of two vertex-disjoint circles and a path meeting each circle at one endpoint and nowhere else. By $m\Gamma$ we mean a graph Γ with every edge replaced by m copies of itself. A *star* S_k is a simple graph having k edges that are all incident to one vertex. Some of these graphs are depicted in Figure 0.2.1.

An *induced subgraph* of Γ is $\Gamma{:}W = (W, E{:}W)$ where $W \subseteq V$ and $E{:}W$ consists of the edges of E whose vertices are contained in W. W is *stable* if $E{:}W = \emptyset$. An *edge-induced subgraph* of Γ is $\Gamma{:}S = \big(V(S), S\big)$ where $S \subseteq E$ and $V(S)$ consists of those vertices that are incident with an edge in S. We let $c(S)$ equal the number of connected components of (V, S).

A graph Γ is *2-connected* (or *inseparable*) if any two edges lie in a common circle.

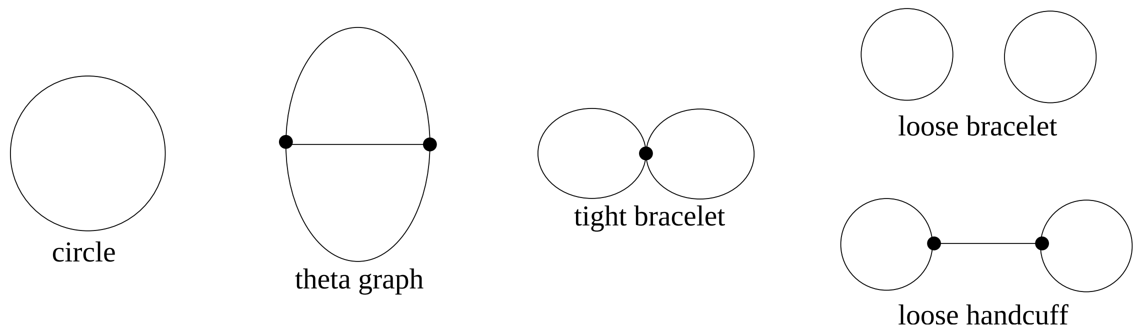


Figure 0.2.1: These graphs are the underlying graphs of the circuits of the bias and lift matroids.

It is *3-connected* if it is 2-connected and has no *2-separation*. A 2-separation is an ordered pair (H, K) of edge-disjoint subgraphs of Γ satisfying: $H \cup K = \Gamma$, H and K have exactly two vertices in common, and H and K each have at least two edges. A *block* of a graph is a maximal inseparable subgraph.

A *homomorphism* (or *map*) of graphs is an incidence-preserving mapping of the vertex and edge sets. That is, a homomorphism $f : \Gamma \to \Gamma'$ consists of mappings $f_V : V \to V'$ and $f_E : E \to E'$ such that $f_V(v)$ and $f_E(e)$ are incident in Γ' if $v \in V$ and $e \in E$ are incident in Γ. A homomorphism is an *isomorphism* if f_V and f_E are bijections and if $v \in V$ and $e \in E$ are incident in Γ if and only if $f_V(v)$ and $f_E(e)$ are incident in Γ'.

A *biased graph* $\Omega = (\Gamma, \mathcal{B})$ consists of a graph Γ and a subclass $\mathcal{B}$ of the class of circles of Γ which satisfies the property: if two circles of a theta graph are in $\mathcal{B}$, then the third circle is also in $\mathcal{B}$. Such a subclass is called a *linear class of circles*. We call $\mathcal{B}$ the set of *balanced circles* of Ω. If writing only $\mathcal{B}$ is ambiguous, we write $\mathcal{B}(\Omega)$ instead. We call Γ the *underlying graph* of Ω. If Γ is not explicitly defined, we write $||\Omega||$ instead. We write $E(\Omega) = E$ and $V(\Omega) = V$.

In a biased graph Ω, we let $U(\Omega) = \{v \in V \mid v \text{ supports an unbalanced loop}\}$. The biased graph $\Omega^\bullet$ denotes Ω with an unbalanced loop added at each vertex in $V \backslash U(\Omega)$. We call Ω *simply biased* if it has no balanced loops, balanced digons, or pairs of unbalanced loops at the same vertex.

If $W \subseteq V$ and $S \subseteq E$, then $\Omega{:}W$ and $\Omega{:}S$ denote the respective subgraphs $\Gamma{:}W$ and $\Gamma{:}S$ with balance of circles the same as in Ω.

The edge set S is *balanced* if every circle in it is balanced and is *contrabalanced* if it contains no balanced circle. We say that Ω is balanced or contrabalanced if its edge set is balanced or contrabalanced, respectively.

We let $b(S)$ equal the number of connected components of (V, S) which are balanced. The *balance-closure* of S, denoted $\operatorname{bcl}(S)$, is the set

$$S \cup \big\{e \in S^c \mid \text{there is a balanced circle } C \text{ such that } e \in C \subseteq S \cup \{e\}\big\}.$$

Two biased graphs Ω_1 and Ω_2 are *isomorphic* when there is a graph isomorphism $f : \Gamma_1 \longrightarrow \Gamma_2$ such that $\mathcal{B}(\Omega_2) = \big\{f_{E_1}(C_1) \mid C_1 \in \mathcal{B}(\Omega_1)\big\}$.

Biased graphs are a combinatorial generalization of *gain graphs*. A gain graph $\Phi = (\Gamma, \phi)$ consists of a graph Γ and a *gain mapping* ϕ from the edges of Γ into a group $\mathfrak{G}$, called the *gain group*. We require that $\phi(e^{-1}) = \phi(e)^{-1}$, where e^{-1} means e with its orientation reversed. Thus $\phi(e)$ depends on the orientation of e, but neither orientation is preferred.

Associated with Φ is a class $\mathcal{B}(\Phi)$ of balanced circles. Let B be a circle of Γ. To decide whether or not B is balanced, choose an edge e_1 of B and a direction (clockwise or counterclockwise) to traverse B. Let $e_1, e_2, \ldots, e_k$ be the edges of B in the order in which they are traversed, and let them be oriented in this direction. The *gain* of B is $\phi(B) = \phi(e_1)\phi(e_2)\cdots\phi(e_k)$. Then $B \in \mathcal{B}(\Phi)$ if $\phi(B) = 1$. Whether or not $B \in \mathcal{B}(\Phi)$ is independent of the choices of e_1 and the direction in which B is traversed.

A gain graph $\Phi = (\Gamma, \phi)$ yields the biased graph $\big(\Gamma, \mathcal{B}(\Phi)\big)$. We denote this biased graph by $\langle\Phi\rangle$. A biased graph is called *gain biased* if it equals $\langle\Phi\rangle$ for some gain graph.

Now we discuss other types of biased graphs. Denote by $\langle\Gamma\rangle$ the biased graph whose underlying graph is Γ, in which every circle is balanced.

Let $\mathcal{B}$ be a set of Hamiltonian circles in Γ, a graph with at least three vertices. We call $(\Gamma, \mathcal{B})$ a *Hamiltonian bias* of Γ.

Let $\mathfrak{G}$ be a group. By $\mathfrak{G}\Gamma$ we mean the gain graph derived from Γ by replacing each edge of Γ by $\#\mathfrak{G}$ new edges, one bearing each possible gain value. We call $\mathfrak{G}\Gamma$ the $\mathfrak{G}$-*expansion* of Γ. We write $\pm\Gamma$ for the sign-group expansion of Γ.

A final example is a combinatorial generalization of a group expansion. A *biased expansion* of Γ is a biased graph Ω together with a *projection* mapping $\pi : ||\Omega|| \to \Gamma$ that is the identity on vertices, is surjective, and has the *Circle Lifting Property*: for each circle $C = e_1 e_2 \cdots e_l$ in Γ and each $\tilde{e}_1 \in \pi^{-1}(e_1), \ldots, \tilde{e}_{l-1} \in \pi^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in \pi^{-1}(e_l)$ for which $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_l$ is balanced. In addition, for each $e \in E$, all digons in $\pi^{-1}(e)$ are unbalanced.

For a graph $\Gamma = (V, E)$, the *graphic matroid* $G(\Gamma)$ is the matroid whose point set is E and whose circuits are the circles of Γ. We frequently refer to $G(\Gamma)$ as the matroid of Γ.

Associated with a biased graph $\Omega = (\Gamma, \mathcal{B})$ are two matroids: the *bias matroid* $G(\Omega)$ and the *lift matroid* $L(\Omega)$. The point set of both matroids is $E(\Omega)$; and each matroid is easily defined in terms of its circuits. The circuits of $G(\Omega)$ are the balanced circles and all contrabalanced theta graphs, tight bracelets, and loose handcuffs. The circuits of $L(\Omega)$ are the balanced circles, together with all contrabalanced theta graphs and tight and loose bracelets. Thus $G(\Gamma)$ coincides with $G\big(\langle\Gamma\rangle\big)$ and $L\big(\langle\Gamma\rangle\big)$.

Chapter 1

Circuit Signatures and Modular Triples

1.1 Introduction

This chapter contains two main theorems that were inspired by the theories of matroids, oriented matroids, and gain graphs. The first, Theorem 1.3.1, uses the notion of *modular triples of circuits* (a generalization of theta graphs) to characterize four types of circuit signatures of matroids. The second main result is the combination of Theorems 1.4.4 and 1.4.5. They say that it is possible to construct elementary lifts of a ternary (Theorem 1.4.4) or binary (Theorem 1.4.5) matroid by labeling its elements with members of certain groups. Our two main results are related insofar as Theorem 1.3.1 is instrumental to the proof of Theorem 1.4.4 and was, in part, conceived for this purpose.

The lift matroids in Theorems 1.4.4 and 1.4.5 arise as a generalization of a lift construction due to Zaslavsky [17, Section 3], which is an application of a lift construction due to Dowling and Kelly [6, Section 6]. These constructions require a special type of class of matroid circuits, called a *linear* class. Given a matroid M and a linear class $\mathcal{B}$ of its circuits, Dowling and Kelly described how $\mathcal{B}$ determines an elementary lift of M. Zaslavsky applied their construction in the case where M is a graphic matroid. He labeled the edges of the associated graph with group elements, called *gains*. The gain of an edge is arbitrarily associated with a fixed orientation of that edge. If the edge is traversed in the opposite direction, then the gain is the inverse of the group element. Zaslavsky defined $\mathcal{B}$, the set of *balanced circles*, to consist of those edge sets of circles for which 1 is the product of the gains of the edges as they are traversed around the circle. In this way, gains are used to construct lifts of graphic matroids.

We want to use gains to construct lifts of more matroids than graphic matroids. This involves labeling the elements of a matroid with gains and defining the notion of a balanced circuit. In order to define a balanced circuit, we need a way to distinguish between the gain that is associated with a matroid element and its inverse. In a graph, this feat is accomplished by direction, as explained above. Our substitute for direction is circuit signatures of matroids. This is appropriate because Zaslavsky's definition of a balanced circle has an interpretation in terms of circuit signatures. Arbitrarily orient both the edges of a graph and its circles. An element of a signed circuit is positive if the orientations of the edge and the circle agree and is negative if they disagree. This defines a circuit signature of the graphic matroid. (This is a special type of circuit signature, called an orientation.) A circle is balanced (has unit gain) precisely when the product of the gains of the positive elements and the inverses of the gains of the negative elements equals 1. This method of defining balanced circles generalizes easily to a definition of balanced circuits for arbitrary matroids.

Given a matroid whose edges are labeled by gains and given a signature of its circuits, let $\mathcal{B}$ be the set of *balanced circuits*, those circuits with unit gain. Dowling and Kelly's lift construction applies only when $\mathcal{B}$ is linear, which inspires the following definition: a *lifting signature* is a circuit signature that forces $\mathcal{B}$ to be linear. Thus gains can be used to lift a matroid if and only if it has a lifting signature. Now we ask, which matroids have lifting signatures? The answer, according to Theorems 1.4.4 and 1.4.5, is the ternary and binary matroids.

To prove that ternary matroids can be lifted using gains, we apply Theorem 1.3.1 and find that ternary signatures [12] (which are a particular kind of circuit signature for ternary matroids) and lifting signatures coincide. I originally proved Theorem 1.3.1 for lifting signatures. Then I found that modular triples also characterize weak orientations [1], orientations [2], and ternary signatures [12]. Our characterization of these circuit signatures in terms of modular triples is new. Many other characterizations already exist, such as by forbidden minors, circuit elimination, and orthogonality. The following chart shows where these characterizations appear in this chapter. The proof of Theorem 1.3.1 employs some aspects of the circuit elimination and forbidden minor characterizations.

	Orientations	Weak Orientations	Ternary/Lifting Signatures
Forbidden minors	Thm. 1.2.3	Thm. 1.2.10	Thm. 1.2.18(4)
Circuit elimination	(4) in Sect. 1.2.4	Thm. 1.2.9	Thm. 1.2.18(2)
Orthogonality	Thm. 1.2.6	Thm. 1.2.7	Thm. 1.2.18(3)
Modular triples	Thm. 1.3.1(2)	Thm. 1.3.1(1)	Thm. 1.3.1(3) and (4)

Though I created Theorem 1.3.1 to help understand lifting signatures, it has other applications. For example, it allows us to provide quick, easy proofs of several known facts about circuit signatures (see Corollary 1.5.1).

The rest of this chapter consists of five sections. Section 1.2 contains background information. In Section 1.3, we use modular triples to characterize weak orientations, orientations, and ternary signatures. The characterization of lifting signatures by modular triples appears in Section 1.4. In that section, we also answer the question:

Which matroids can be lifted using gains? In Section 1.5, we give some applications of our main theorems. Finally, in Section 1.6, we pose some questions for future consideration.

1.2 Background

1.2.1 Using Linear Classes of Circuits to Construct Matroid Lifts

Let M be a matroid with ground set E. In [6], Dowling and Kelly explain how to use a linear class of circuits to construct an elementary lift of M. In this section, we discuss the construction, but we use different terminology. Most of our notation and terminology is from [11], which is a user-friendly reference for matroid theory.

Modular Triples

We say that (C_1, C_2, C_3) is a *modular triple of circuits* of M if the three circuits are distinct and, for distinct i, j, and k, $C_k \subseteq C_i \cup C_j$ and (C_i, C_j) is a modular pair. $\big($Recall that (C_i, C_j) is a *modular pair of circuits* if $r(C_i \cup C_j) = |C_i \cup C_j| - 2.\big)$ A generalization of this concept appears in [1, Section 5], and our definition is equivalent to the one in [4, Section 7.1]. If M is a graphic matroid, then a modular triple of circuits consists of the three circles of a theta graph. Thus a modular triple of circuits is a matroid generalization of a theta graph.

We say that (H_1, H_2, H_3) is a *modular triple of copoints* of M if the three copoints are distinct and intersect in a coline. (Copoints and colines are flats whose ranks are less than the rank of M by 1 and 2, respectively.) In other words, for distinct i, j, and k, $H_i \cap H_j \subseteq H_k$ and (H_i, H_j) is a modular pair. $\big($Recall that (F_i, F_j) is a *modular* *pair of flats* if $r(F_i) + r(F_j) = r(F_i \cup F_j) + r(F_i \cap F_j)$.) This definition appears in [4, Section 7.1].

It is not hard to show that (C_1, C_2, C_3) is a modular triple of circuits of M if and only if $(E\backslash C_1, E\backslash C_2, E\backslash C_3)$ is a modular triple of copoints of M^*. We usually write H_i^* instead of $E\backslash C_i$. If L^* is the dual coline (i.e., in M^*) at which H_1^*, H_2^*, and H_3^* meet, then $L^* = E\backslash(C_1 \cup C_2 \cup C_3)$. Figure 1.2.1 is helpful when thinking about modular triples. An easy, yet important, observation is that I_{13}, I_{12}, and I_{23} are nonempty (because the three circuits are distinct).

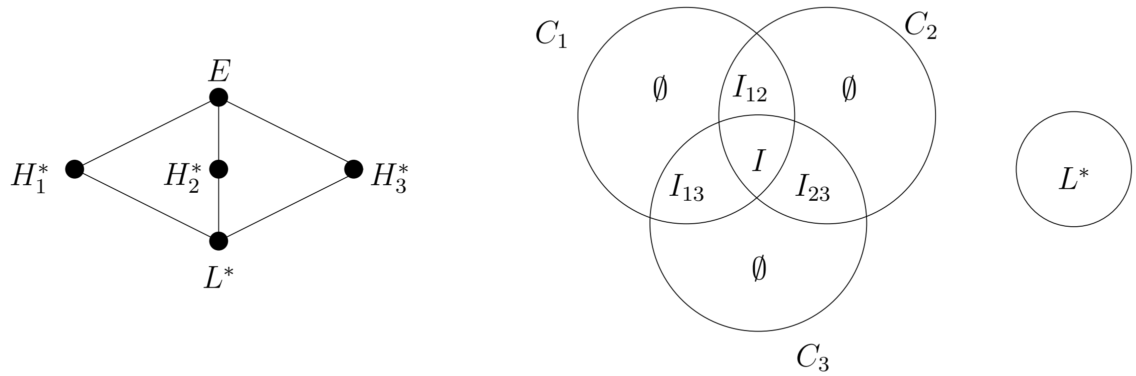


Figure 1.2.1: On the left is a portion of the lattice of flats of M^* in which (H_1^*, H_2^*, H_3^*) is a modular triple of copoints. On the right we see (C_1, C_2, C_3), the corresponding modular triple of circuits of M.

Throughout this chapter, whenever (C_1, C_2, C_3) is a modular triple of circuits, the sets I, I_{13}, I_{12}, and I_{23} are defined as in Figure 1.2.1.

At this point, we include two useful lemmas about modular triples of circuits.

Lemma 1.2.1. *A matroid M is binary if and only if, for each modular triple (C_1, C_2, C_3) of circuits, $C_1 \cap C_2 \cap C_3 = \emptyset$.*

Proof. Let (C_1, C_2, C_3) be a modular triple of circuits of M, and suppose $w \in C_1 \cap C_2 \cap C_3$. Thus $w \notin H_1^* \cup H_2^* \cup H_3^*$. Also, let L^* be the coline at which these copoints meet, so $w \notin L^*$. Since the copoints in the interval $[L^*, E]$ partition the elements of $E\backslash L^*$, there exists another copoint in this interval, say H_4^*, that contains w. But then the lattices of flats of $M^*|(H_1^* \cup H_2^* \cup H_3^* \cup H_4^*)/L^*$ and $U_{2,4}$ are isomorphic, which means that M^* has a $U_{2,4}$ minor. Equivalently, M has a $U_{2,4}$ minor.

Now assume that M is not binary. Thus M^* has a $U_{2,4}$ minor. By the Scum Theorem, M^* has a coline L^* such that $[L^*, E]$ contains four distinct copoints, namely H_1^*, H_2^*, H_3^*, and H_4^*. Since (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of M^*, (C_1, C_2, C_3) is a modular triple of circuits of M. Let $w \in (H_4^* \backslash L^*)$, so $w \notin H_1^* \cup H_2^* \cup H_3^*$. Thus $w \in C_1 \cap C_2 \cap C_3$. □

Lemma 1.2.2. *Let (C_1, C_2) be a modular pair of circuits of a matroid M, and let $e \in C_1 \cap C_2$. Then there exists a unique circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2)\backslash\{e\}$. Moreover, (C_1, C_2, C_3) is a modular triple of circuits of M.*

Proof. We know that (H_1^*, H_2^*) is a modular pair of copoints of M^*. Assume they meet at the coline L^*. Since $e \in C_1 \cap C_2$, it follows that $e \notin H_1^* \cup H_2^*$. Since the copoints in the interval $[L^*, E]$ partition the elements of $E \backslash L^*$, there exists a third copoint in this interval, say H_3^*, that contains e. Thus $e \notin C_3$. Since (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of M^*, (C_1, C_2, C_3) is a modular triple of circuits of M. Therefore, $C_3 \subseteq (C_1 \cup C_2)$.

Suppose $C_3' \subseteq (C_1 \cup C_2)\backslash\{e\}$. Then it corresponds to a copoint in the interval $[L^*, E]$ that contains e. But there is exactly one such copoint, namely H_3^*. Thus $C_3' = C_3$, which proves uniqueness. □

Linear Subclasses

Let $\mathcal{B}$ be a subclass of circuits of M. If, for each modular triple of circuits, either 0, 1, or 3 of these circuits are in $\mathcal{B}$, we say that $\mathcal{B}$ is a *linear subclass of circuits* of M. An equivalent definition appears in [17, Section 3]. It is a matroid generalization of a linear subclass of circles, which was introduced in Section 0.2.

Similarly, let $\mathcal{H}$ be a subclass of copoints of M. If, for each modular triple of copoints, either 0, 1, or 3 of these copoints are in $\mathcal{H}$, we say that $\mathcal{H}$ is a *linear subclass of copoints* of M. Crapo gave this definition in [5, Section 6].

It is easy to show that $\mathcal{B}$ is a linear subclass of circuits of M if and only if $\{E\backslash C : C \in \mathcal{B}\}$ is a linear subclass of copoints of M^*.

My definitions of linear subclasses of circuits and copoints are different from those in the literature because they are stated in terms of modular triples.

Single-Element Extensions and Modular Cuts

Let M and N be matroids. We say that N is a *single-element extension of* M if $M = N\backslash\{e\}$.

There is a one-to-one correspondence between single-element extensions and modular cuts of M [11, Section 7.2]. A set $\mathcal{M}$ of flats of M is called a *modular cut* if it satisfies the following rules:

(1) if $F \in \mathcal{M}$ and F' is a flat containing F, then $F' \in \mathcal{M}$, and

(2) if F_1, $F_2 \in \mathcal{M}$ and (F_1, F_2) is a modular pair of flats, then $F_1 \cap F_2 \in \mathcal{M}$.

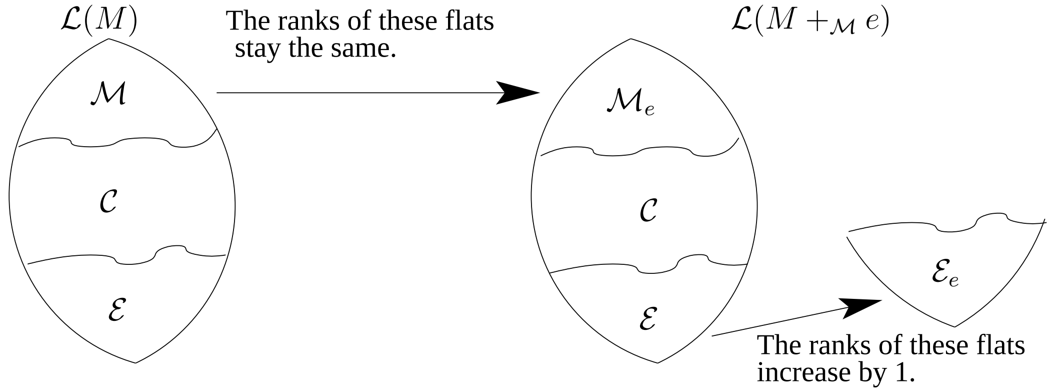
If $M = N\backslash\{e\}$ and $\mathcal{M}$ is the modular cut corresponding to the single-element extension N, we write N as $M +_{\mathcal{M}} e$. Denote the set of flats of M by $\mathcal{F}(M)$. Figure 1.2.2 illustrates how $\mathcal{F}(M +_{\mathcal{M}} e)$ is related to $\mathcal{F}(M)$. In this figure,

$$\mathcal{C} = \{F \in \mathcal{F}(M) : F \notin \mathcal{M} \text{ and } F \text{ is covered by an element of } \mathcal{M}\},$$

and

$$\mathcal{E} = \mathcal{F}(M)\backslash(\mathcal{M} \cup \mathcal{C}).$$

Figure 1.2.2: This figure illustrates the close connection between $\mathcal{L}(M)$, the lattice of flats of M, and $\mathcal{L}(M +_{\mathcal{M}} e)$, the lattice of flats of $M +_{\mathcal{M}} e$. Here, $\mathcal{M}_e = \{F \cup \{e\} : F \in \mathcal{M}\}$; and $\mathcal{E}_e$ is defined similarly.



Now it is easy to see that

$$r(M +_{\mathcal{M}} e) = \begin{cases} r(M) + 1 & \text{if } \mathcal{M} = \emptyset, \\ r(M) & \text{otherwise.} \end{cases}$$

Also, $\mathcal{M} = \emptyset$ if and only if e is a coloop in $M +_{\mathcal{M}} e$, and $\mathcal{M} = \mathcal{F}(M)$ if and only if e is a loop in $M +_{\mathcal{M}} e$.

We say that a modular cut $\mathcal{M}$ of M is *nontrivial* if $\mathcal{M} \neq \emptyset$ and $\mathcal{M} \neq \mathcal{F}(M)$.

Elementary Lifts

A matroid L is an *elementary lift* of M if there exists a matroid N and an element e of N that is not a loop or coloop such that $N/\{e\} = M$ and $N\backslash\{e\} = L$. Since N is a single-element extension of L, $N = L +_{\mathcal{M}} e$. So if L is an elementary lift of M, then $M = (L +_{\mathcal{M}} e)/\{e\}$ for some nontrivial modular cut $\mathcal{M}$ of L. It is easy to see that the rank of M is one less than that of L and that their ground sets are the same.

We are ready to show how a linear subclass of circuits of M enables the construction of a unique elementary lift of M. Let $\mathcal{B}$ be such a subclass. Define

$\mathcal{B}^* = \{E\backslash C : C \in \mathcal{B}\}$, so $\mathcal{B}^*$ is a linear subclass of copoints of M^*. Then

$$\mathcal{M}_0 = \{F \in \mathcal{F}(M^*) : \text{ every copoint containing } F \text{ is in } \mathcal{B}^*\} \tag{1.2.1}$$

is a modular cut of M^* (see [5, Section 6]). This modular cut is nontrivial as long as $\mathcal{B}$ does not contain all circuits of M. (It is impossible for $\mathcal{M}_0$ to be empty because $\mathcal{M}_0$ always contains E.) Assume that there is a circuit of M that is not in $\mathcal{B}$. Then M^* is an elementary lift of $\big(M^* +_{\mathcal{M}_0} e\big)/\{e\}$. Equivalently, $\big((M^* +_{\mathcal{M}_0} e)/\{e\}\big)^*$ is an elementary lift of M. We denote this elementary lift by $L(M, \mathcal{B})$. Different choices of $\mathcal{B}$ yield different elementary lifts of M.

1.2.2 Using Gains to Lift Graphic Matroids

Let Φ be the gain graph in Figure 1.2.3. In this example,

$$\mathcal{B}(\Phi) = \big\{\{a, e, d\}, \{d, f, c\}, \{a, e, f, c\}\big\}.$$

The balanced circles are the three circles of a theta graph. This is no coincidence;

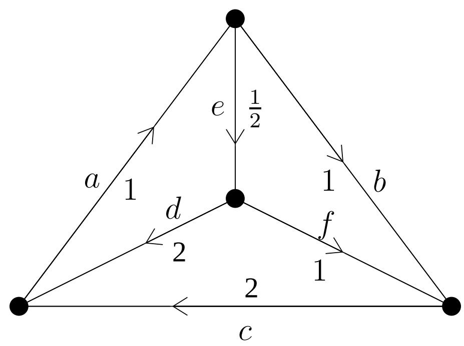


Figure 1.2.3: This is a gain graph with gain mapping $\phi : E(K_4) \to \mathbb{R}^*$.

the set of balanced circles of a gain graph is a linear subclass of circuits of the graphic matroid $G(\Gamma)$ [16, Proposition 5.1]. It follows from Section 1.2.1 that $L\big(G(\Gamma), \mathcal{B}(\Phi)\big)$ is an elementary lift of $G(\Gamma)$ (unless all circles are balanced). In [17, Section 3], Zaslavsky refers to this lift as the *lift of* $G(\Gamma)$ *along* $\mathcal{B}(\Phi)$. We also denote it by $L(\Phi)$. An easier description of $L(\Phi)$ is given in Section 0.2.

If Φ is the gain graph in Figure 1.2.3, then the circuits of $L(\Phi)$ are

$$\{a, e, d\}, \{d, f, c\}, \text{ and } \{a, e, f, c\}.$$

$\big($In this example, the balanced circles of Φ happen to coincide with the circuits of $L(\Phi)$.$\big)$ The description of $L(\Phi)$ in Section 0.2 makes this easy to see. The description in Section 1.2.1 yields the same lift matroid, but the construction is more involved.

1.2.3 Circuit Signatures

A *signed set* is a set X together with an ordered bipartition of X into subsets X^+ and X^-. We denote the signed set by the ordered pair (X^+, X^-). The signed set has *underlying set* (or *support*) X. We denote both the signed set and its underlying set by X. Context indicates whether we are dealing with sets or signed sets.

If a signed set X has empty support, we say that $X = \emptyset$.

Sometimes, instead of denoting a signed set by $\big(\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_n\}\big)$, we write it as $a_1 \cdots a_p \overline{b_1} \cdots \overline{b_n}$.

If X is a signed subset of E, then for each $e \in E$, define

$$X(e) = \begin{cases} +1 & \text{if } e \in X^+, \\ -1 & \text{if } e \in X^-, \\ 0 & \text{if } e \notin X^+ \cup X^-. \end{cases}$$

For every signed set X, we define the *negative* of X, denoted by $-X$, to be the signed set (X^-, X^+). We write $X = \pm Y$ if $X = Y$ or $X = -Y$. Also, $X \backslash T$ is the signed set $(X^+\backslash T, X^-\backslash T)$.

Let $\mathcal{C}$ be a collection of signed sets. We define $\mathrm{Min}(\mathcal{C})$ to be the collection of elements of $\mathcal{C}$ with setwise minimal support.

Let M be a matroid on E, and let $\mathcal{C}$ be a collection of signed subsets of E. We say that $\mathcal{C}$ is a *circuit signature* of M if:

1. every signed set in $\mathcal{C}$ has a circuit of M as underlying set, and
2. for every circuit C of M, there are precisely two members of $\mathcal{C}$ with underlying set C, and these two signed sets are negatives of each other.

Frequently, when we write a circuit signature of M, we only specify one of the two signed sets with underlying set C for each circuit C of M.

Let $\mathcal{C}$ be a circuit signature of M. Define $-\mathcal{C} = \{-X \mid X \in \mathcal{C}\}$. Additional circuit signatures of M can be created by the process of reorientation. Given a signed set X in $\mathcal{C}$ and a set $A \subseteq E$, the *reorientation of* X *on* A, denoted by ${}_{\overline{A}}X$, is the signed set derived from X by reversing the signs of the elements of A. Technically,

$$ {}_{\overline{A}}X^+ = \big(X^+\backslash A\big) \cup \big(X^- \cap A\big) $$

and

$$ {}_{\overline{A}}X^- = \big(X^-\backslash A\big) \cup \big(X^+ \cap A\big). $$

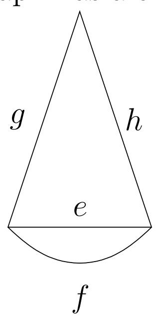
The *reorientation of* $\mathcal{C}$ *on* A is the circuit signature ${}_{\overline{A}}\mathcal{C} = \{{}_{\overline{A}}X : X \in \mathcal{C}\}$.

Let $e \in E$. Define $\mathcal{C}\backslash e = \{X \in \mathcal{C} : e \notin X\}$, the *deletion of* $\mathcal{C}$ *by* e. Also define the *contraction of* $\mathcal{C}$ *by* e, denoted by $\mathcal{C}/e$, to be the set $\mathrm{Min}\big\{X\backslash\{e\} : X \in \mathcal{C} \text{ and } X\backslash\{e\} \neq \emptyset\big\}$. Every collection obtained from $\mathcal{C}$ by a succession of deletions and contractions is called a *minor* of $\mathcal{C}$.

It is easy to see that $\mathcal{C}\backslash e$ and $\mathcal{C}/e$ are circuit signatures of $M\backslash e$ and M/e, respectively. Hence any minor of $\mathcal{C}$ is a circuit signature of the associated matroid minor. However, the order in which elements are deleted and contracted from $\mathcal{C}$ may affect the resulting minor. Consider $\mathcal{C} = \{eg\overline{h}, fgh, ef\}$, a circuit signature of the matroid of the graph in Figure 1.2.4. Then $\mathcal{C}/e/f = \{g\overline{h}\}$, but $\mathcal{C}/f/e = \{gh\}$.

Let S and T be disjoint subsets of E. If, for each choice of S and T, the minor of $\mathcal{C}$ that results after deleting the elements of S and contracting the elements of T is independent of the order in which the elements are deleted and contracted, then we say that $\mathcal{C}$ is *minorable.* Thus, in our example, $\mathcal{C}$ is not minorable.

Figure 1.2.4: The matroid of this graph has a circuit signature that is not minorable.



Suppose that $\mathcal{C}$ is a circuit signature of M, and let C_1, C_2, and C_3 be elements of $\mathcal{C}$. We say that (C_1, C_2, C_3) is a *modular triple of signed circuits* if, as underlying sets, (C_1, C_2, C_3) is a modular triple of circuits of M. Make a similar definition for a *modular pair of signed circuits.*

1.2.4 Orientations of Matroids

Bland and Las Vergnas introduced oriented matroids in [2]. Just as matroids can be defined in many cryptomorphic ways, oriented matroids can be defined in terms of circuits, cocircuits, vectors, covectors, and more. We are most interested in the signed circuits of an oriented matroid, which we now define. A collection $\mathcal{C}$ of signed subsets of a set E is the set of *signed circuits* of an oriented matroid on E if it satisfies the following axioms:

1. $\emptyset \notin \mathcal{C}$,

2. (*Symmetry*) $\mathcal{C} = -\mathcal{C}$,

3. (*Incomparability*) for all X, $Y \in \mathcal{C}$, if the support of X is contained in the support of Y, then $X = \pm Y$, and

4. (*Elimination*) for all $X, Y \in \mathcal{C}$ such that $X \neq -Y$, and all $e \in X^+ \cap Y^-$, there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq \big(X^+ \cup Y^+\big) \backslash \{e\}$ and $Z^- \subseteq \big(X^- \cup Y^-\big) \backslash \{e\}$.

If M is a matroid on E with circuit signature $\mathcal{C}$, it is clear that $\mathcal{C}$ determines (is the set of signed circuits of) an oriented matroid if $\mathcal{C}$ also satisfies (4). In this case, we say that $\mathcal{C}$ is an *orientation* of M and that M is *orientable*. Las Vergnas showed that (4) only needs to be verified for modular pairs of signed circuits (see [10, Theorem 2.1]).

We can use forbidden minors to determine whether or not a circuit signature of a matroid is an orientation.

Theorem 1.2.3 ([9, Theorem 2.1]). *Let M be a matroid and let $\mathcal{C}$ be a minorable circuit signature of M. Then $\mathcal{C}$ is an orientation of M if and only if $\mathcal{C}$ has no minor isomorphic to a reorientation of the signature $\{12, 13, 23\}$ of $U_{1,3}$ or a reorientation of the signature $\big\{12\overline{3}, 1\overline{24}, 134, 23\overline{4}\big\}$ of $U_{2,4}$.*

1.2.5 Weak Orientations of Matroids

Weakly oriented matroids were introduced by Bland and Jensen in [1]. Matroids that are weakly orientable are an intermediate class between all matroids and orientable matroids. Though we will eventually be concerned with a characterization of weakly oriented matroids in terms of circuit signatures, we first define them as Bland and Jensen did in [1], in terms of the Minty Coloring Property. This definition shows that weakly oriented matroids arise naturally when studying matroids and oriented matroids.

Let E be a finite set. When we write $(E, \mathcal{C}, \mathcal{C}^*)$, it is understood that $\mathcal{C}$ and $\mathcal{C}^*$ are collections of signed subsets of E. When we say that $(E, \mathcal{C}, \mathcal{C}^*)$ *satisfies* $(\star)$, we mean that neither $\mathcal{C}$ nor $\mathcal{C}^*$ contains the empty set and that both satisfy symmetry and incomparability (as defined in Section 1.2.4).

Define a *3-coloring* of E to be a triple (R, B, W) of subsets that partition E. Given a 3-coloring (R, B, W) of E, we say that $(E, \mathcal{C}, \mathcal{C}^*)$ satisfies the *Minty Coloring Property* (MCP) at (R, B, W) if for every $e \in R$ exactly one of the following two statements holds:

$$\text{there exists } X \in \mathcal{C} \text{ such that } e \in X^+ \subseteq R \cup B \text{ and } X^- \subseteq B$$

or

$$\text{there exists } Y \in \mathcal{C}^* \text{ such that } e \in Y^+ \subseteq R \cup W \text{ and } Y^- \subseteq W.$$

The next two theorems are stated in [1]. Theorem 1.2.4 is a restatement of a result of Minty, and Theorem 1.2.5 is a sharpening of a result of Bland and Las Vergnas.

Theorem 1.2.4 ([1, Theorem 1.4])**.** *Assume that $(E, \mathcal{C}, \mathcal{C}^*)$ satisfies $(\star)$. Then $(E, \mathcal{C}, \mathcal{C}^*)$ determines a dual pair of matroids if and only if the MCP is satisfied by $(E, \mathcal{C}, \mathcal{C}^*)$ at all 3-colorings (R, B, W) of E having $|R| = 1$.*

To say that $(E, \mathcal{C}, \mathcal{C}^*)$ *determines a dual pair of matroids* means that there is a matroid on E whose set of circuits is the collection of supports of elements of $\mathcal{C}$, and whose set of cocircuits is the collection of supports of elements of $\mathcal{C}^*$.

Theorem 1.2.5 ([1, Theorem 1.5])**.** *Assume that $(E, \mathcal{C}, \mathcal{C}^*)$ satisfies $(\star)$. Then $(E, \mathcal{C}, \mathcal{C}^*)$ determines a dual pair of oriented matroids if and only if the MCP is satisfied by $(E, \mathcal{C}, \mathcal{C}^*)$ at all 3-colorings (R, B, W) of E having $1 \leq |R| \leq 3$.*

To say that $(E,\mathcal{C},\mathcal{C}^*)$ *determines a dual pair of oriented matroids* means that there is an oriented matroid on E whose set of signed circuits is $\mathcal{C}$, and whose set of signed cocircuits is $\mathcal{C}^*$.

The original (and equivalent) version of Theorem 1.2.5 has "$1 \le |R| \le 3$" replaced by "$1 \le |R|$" [3, Theorem 1].

Now it is clear how weakly oriented matroids should be defined. Assume that $(E,\mathcal{C},\mathcal{C}^*)$ satisfies $(\star)$. We say that $(E,\mathcal{C},\mathcal{C}^*)$ *determines a dual pair of weakly oriented matroids* if the MCP is satisfied by $(E,\mathcal{C},\mathcal{C}^*)$ at all 3-colorings (R,B,W) of E having $1 \le |R| \le 2$. We say that $\mathcal{C}$ is the *set of signed circuits* of a weakly oriented matroid $\mathcal{W}$ on E, and $\mathcal{C}^*$ is the *set of signed cocircuits* of $\mathcal{W}$. By Theorem 1.2.4, the collection of supports of elements of $\mathcal{C}$ is the set of circuits of a matroid, call it M. We say that $\mathcal{C}$ is a *weak orientation* of M and that M is *weakly orientable.* We also say that $(E,\mathcal{C})$ *determines a weakly oriented matroid* if there is a matroid on E for which $\mathcal{C}$ is a weak orientation.

We can also understand weakly oriented matroids through an orthogonality condition. We say that signed subsets X and Y of E are *orthogonal* if

$$\big(X^+ \cap Y^+\big) \cup \big(X^- \cap Y^-\big) \neq \emptyset \Longleftrightarrow \big(X^+ \cap Y^-\big) \cup \big(X^- \cap Y^+\big) \neq \emptyset.$$

Bland and Las Vergnas related the orthogonality condition to orientability in [2, Theorem 2.2].

Theorem 1.2.6. *Let $\mathcal{C}$ be a circuit signature of a matroid M on E and let $\mathcal{C}^*$ be a circuit signature of M^*. $(E,\mathcal{C},\mathcal{C}^*)$ determines a dual pair of oriented matroids if and only if the orthogonality condition holds for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}^*$ such that $2 \le |X \cap Y| \le 3$.*

Now the following theorem is not surprising.

Theorem 1.2.7 ([1, Theorem 1.10]). *Let $\mathcal{C}$ be a circuit signature of a matroid M on E and let $\mathcal{C}^*$ be a circuit signature of M^*. $(E, \mathcal{C}, \mathcal{C}^*)$ determines a dual pair of weakly oriented matroids if and only if the orthogonality condition holds for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}^*$ having $|X \cap Y| = 2$.*

Bland and Jensen also give a characterization of weak orientations by circuit elimination.

Theorem 1.2.8 ([1, Theorem 6.1]). *Given a sign-symmetric collection $\mathcal{C}$ of signed subsets of E, $(E, \mathcal{C})$ determines a weakly oriented matroid if and only if*

(E0) $\mathcal{C}$ is a collection of incomparable, nonempty subsets of E;

(E1) For every X_1, $X_2 \in \mathcal{C}$ with $e \in X_1^+ \cap X_2^-$ and $X_1 \neq -X_2$,

- *(i) if $f \in \big(X_1^+ \backslash X_2^-\big) \cup \big(X_1^- \backslash X_2^+\big)$, then there exists $X_3 \in \mathcal{C}$ with $f \in X_3 \subseteq \big(X_1 \cup X_2\big) \backslash \{e\}$;*
- *(ii) there are $e_1 \in X_1 \backslash X_2$, $e_2 \in X_2 \backslash X_1$, and $X_4 \in \mathcal{C}$ satisfying $X_4 \subseteq \big(X_1 \cup X_2\big) \backslash \{e\}$ so that $X_4(e_1)X_4(e_2) = X_1(e_1)X_2(e_2)$.*

Axioms (E0) and (E1.i) imply the strong elimination axiom for matroids. This is the only nontrivial detail in showing that Theorem 1.2.8 can be stated in the more compact form given in Theorem 1.2.9.

Theorem 1.2.9. *Let $\mathcal{C}$ be a circuit signature of a matroid M. Then $\mathcal{C}$ is a weak orientation of M if and only if for every X_1, $X_2 \in \mathcal{C}$ with $e \in X_1^+ \cap X_2^-$ and $X_1 \neq -X_2$,*

(i) if $f \in \big(X_1^+ \backslash X_2^-\big) \cup \big(X_1^- \backslash X_2^+\big)$, then there exists $X_3 \in \mathcal{C}$ with $f \in X_3 \subseteq (X_1 \cup X_2) \backslash \{e\}$; and

(ii) there are $e_1 \in X_1 \backslash X_2$, $e_2 \in X_2 \backslash X_1$, and $X_4 \in \mathcal{C}$ satisfying $X_4 \subseteq (X_1 \cup X_2) \backslash \{e\}$ so that $X_4(e_1)X_4(e_2) = X_1(e_1)X_2(e_2)$.

As with orientations, weak orientations can be characterized by forbidden minors.

Theorem 1.2.10 ([8, Theorem 1, p. 173])**.** *A circuit signature $\mathcal{C}$ of a matroid is a weak orientation of M if and only if $\mathcal{C}$ has no minor isomorphic to a reorientation of the signature* $\{12, 13, 23\}$ *of* $U_{1,3}$.

1.2.6 Ternary Signatures of Matroids

In order to introduce ternary signatures, we first need to discuss chain groups and their associated matroids. This theory was developed by Tutte. We use terminology, notation, and theorems from [14, Chapter 8] and [15, Section 9.4].

Chain-Group Matroids

Let F be a field, and let S be a finite set. A *chain on S to F* is a map $f : S \to F$. Denote the collection of all chains on S to F by F^S. The *support* of a chain f, denoted by $||f||$, is the set $\{e \in S : f(e) \neq 0\}$.

A *chain group on S to F* is a subspace of F^S.

Let N be a chain group on S to F. A chain f is *elementary* if its support is nonempty and, for all nonzero chains g of N with $||g|| \subseteq ||f||$, we have $||g|| = ||f||$.

Given a chain group N on S to F, there is a corresponding matroid $M(N)$ called the *chain-group matroid* of N. The circuits of $M(N)$ are the supports of the elementary chains of N (see [15, Section 9.4]). (Theorem 1 in [15, Section 9.4] is false. It incorrectly describes the dependent sets of $M(N)$. The proof, however, concentrates on describing the circuits of $M(N)$, and this description is correct.)

The class of chain-group matroids on S to F is exactly the class of matroids on S which are linearly representable over F. We sketch a part of the proof for later reference.

Theorem 1.2.11 ([15, Section 9.4, Theorem 2]). *A matroid M is isomorphic to the chain-group matroid of a chain group over a field F if and only if M is representable over F.*

Sketch of the proof of sufficiency. Let M be a matroid on S, and let $\phi : S \to V$ represent M in V, a vector space over F. Then $M \cong M(N)$ where N is the kernel of the map $\alpha : F^S \to V$ defined by

$$\alpha(f) = \sum_{e \in S} f(e)\phi(e).$$

□

Ternary Signatures

Now we use Tutte's theory of chain groups to construct circuit signatures for ternary matroids, signatures which we later call ternary signatures. We first need Lemma 1.2.13, whose proof depends on the following result.

Lemma 1.2.12 ([11, Proposition 2.2.23]). *Let $[I_r \,|\, D]$ be an $r \times n$ matrix over a field F where $1 \le r \le n-1$. Then the orthogonal complement in F^n of the row space of $[I_r \,|\, D]$ is the row space of $\big[-D^T \mid I_{n-r}\big]$.*

Let M be a matroid that is representable over F. This means that there exists a *representation matrix* whose entries are in F and whose columns are labeled by the elements of E such that a set is independent in M if and only if the associated columns of the matrix are linearly independent.

Lemma 1.2.13. *Let M be a matroid that is representable over F. Let N be the row space of a representation matrix of M^*. Then $M \cong M(N)$.*

Proof. Assume that M is a rank r matroid on a set of size n. Let A be a representation matrix of M^*, and let N be the row space of A. We may assume that the last $n-r$ columns of A are linearly independent. (Otherwise, we could permute columns.) Thus we can apply elementary row operations to reduce A to the matrix $[D \,|\, I_{n-r}]$, whose row space is also N. It follows that $\left[I_r \,|\, -D^T\right]$ is a representation matrix of M over F $\big($see [11, Theorem 2.2.8]$\big)$. Denote the columns of this matrix by $\phi(s_1), \ldots, \phi(s_n)$.

We need to show that $M \cong M(N)$. From the sketch of the proof of Theorem 1.2.11, it suffices to show that the row space of $[D \,|\, I_{n-r}]$ is the kernel of the map $\alpha : F^S \to V$ defined by

$$\alpha\big((g_1, \ldots, g_n)\big) = \sum_{i=1}^{n} g_i \phi(s_i).$$

Assume that $1 \leq r \leq n-1$. By applying Lemma 1.2.12, we can instead prove that the orthogonal complement of the row space of $\left[I_r \,|\, -D^T\right]$ is the kernel of α.

Let $(g_1, \ldots, g_n)$ be in the orthogonal complement of the row space of $\left[I_r \,|\, -D^T\right]$. By definition, $\alpha\big((g_1, \ldots, g_n)\big)$ is an $n \times 1$ column vector whose ith component is the dot product of $(g_1, \ldots, g_n)$ and the ith row of $\left[I_r \,|\, -D^T\right]$. Thus $\alpha\big((g_1, \ldots, g_n)\big) = 0$. Equivalently, $(g_1, \ldots, g_n) \in \ker \alpha$.

If $r = n$, then $M \cong U_{n,n}$ and $M^* \cong U_{0,n}$. The row space of any representation of M^* consists only of 0, and M can be represented by I_n. Thus α is the identity map, so $\ker \alpha = \{0\}$.

The proof for $r = 0$ is similar to the proof for $r = n$. □

Recall that the circuits of M are the supports of the elementary vectors of N.

Now we restrict Lemma 1.2.13 to the case in which $F = \mathrm{GF}(3)$. This is discussed by Roudneff and Wagowski in [12]. $\big($They call N the *Tutte representation* of M over $GF(3)$.$\big)$ The elementary vectors of N can be used to obtain a circuit signature of M: an element s in a signed circuit is positive if the value in the s-coordinate of the elementary vector is +1, and it is negative otherwise. This circuit signature is called a *ternary signature* of M.

Example 1.2.14. As an example, consider the representation

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

of $U_{2,4}$ over GF(3). The elementary vectors of the row space of this representation are

$$(1 \quad 1 \quad -1 \quad 0), \quad (1 \quad -1 \quad 0 \quad -1), \quad (1 \quad 0 \quad 1 \quad 1), \quad (0 \quad 1 \quad 1 \quad -1),$$

and their negatives. This gives the ternary signature $\left\{12\overline{3}, 1\overline{2}\overline{4}, 134, 23\overline{4}\right\}$ of $U_{2,4}$.

The Uniqueness of Ternary Signatures

In Example 1.2.14, it initially appears that a ternary signature of M depends on the representation matrix of M^*. However, Corollary 1.2.16 says that a ternary matroid has a "unique" representation over GF(3) (the word "unique" will be defined shortly). Theorem 1.2.17 shows that the effect of this uniqueness is that the ternary signature is unique up to reorientation. Though the uniqueness of ternary signatures is certainly known, we include a proof due to the lack of a reference where this result is explicitly stated.

Certainly, a representation matrix of a matroid cannot be unique. Multiplying any column by a scalar is a different representation matrix, strictly speaking. So we must define what it means for a matroid to be uniquely F-representable. We use the terminology of Brylawski and Lucas that is found in [7, Sections 2 and 3].

Let M be a rank r matroid with cardinality n that is representable over a field F, and let A_1 and A_2 be representation matrices of M. We define A_1 and A_2 to be *projectively equivalent* if A_2 can be obtained from A_1 by a sequence of operations of the five types listed below.

1. Add a scalar multiple of one row to another.
2. Interchange two rows.
3. Multiply a row by a non-zero member of F.
4. Remove or add a zero row.
5. Multiply a column by a non-zero member of F.

We say that M is *uniquely F-representable* if it can be represented by an $r \times n$ matrix over F and all such matrices are projectively equivalent.

Lemma 1.2.15 ([7, Theorem 3.2])**.** *Let M be a matroid and let $[I \mid A_1]$ and $[I \mid A_2]$ be two representation matrices of M over a field F such that every entry of A_1 and A_2 is 0, 1, or -1. Then $[I \mid A_1]$ and $[I \mid A_2]$ are projectively equivalent.*

We easily get the following corollary to Lemma 1.2.15.

Corollary 1.2.16. *Ternary matroids are uniquely GF(3)-representable.*

Theorem 1.2.17. *A ternary matroid has a unique ternary signature, up to reorientation.*

Proof. Let M be a rank r ternary matroid on n elements. By definition, a ternary signature is determined by the row space of a representation matrix of M^*. Of course, M^* is ternary too. Let A_1 and A_2 be GF(3)-representations of M^*. By Corollary 1.2.16, A_1 and A_2 are projectively equivalent.

If A_1 and A_2 differ by row operations of types (1)–(4), they have the same row space, so the corresponding ternary signatures of M are the same.

Suppose that A_1 and A_2 differ by an operation of type (5), and assume that this column is labeled by element e. We may assume that A_1 and A_2 have r rows; otherwise we could apply row operations of types (1)–(4) to get matrices that have r rows and are row equivalent to A_1 and A_2. Since the field is GF(3), we need only be concerned about when the columns of A_1 and A_2 that are labeled by e are negatives of each other. Suppose $\{s_1, \ldots, s_k\}$ is a circuit of the chain-group matroid M(row space of A_1). So this row space contains some elementary chain, say

$$v = \alpha_1 \cdot (\text{row 1 of } A_1) + \cdots + \alpha_r \cdot (\text{row } r \text{ of } A_1),$$

whose support is $\{s_1, \ldots, s_k\}$. The vector

$$w = \alpha_1 \cdot (\text{row 1 of } A_2) + \cdots + \alpha_r \cdot (\text{row } r \text{ of } A_2)$$

differs from v only in the coordinate labeled by e, and the entry in this coordinate differs by a multiplier of -1. So w has support $\{s_1, \ldots, s_k\}$ also. Moreover, $\{s_1, \ldots, s_k\}$ is a circuit in M(row space of A_2) because w must be elementary. Thus the chain-group matroids M(row space of A_1) and M(row space of A_2) are identical, and the corresponding ternary signatures of M differ by a reorientation of e.

The complete proof follows by induction on the number of operations of types (1)–(5) by which A_1 and A_2 differ. □

Characterizing Ternary Signatures

Since we are interested in circuit signatures, it is convenient that Roudneff and Wagowski characterize ternary signatures by a signed circuit elimination axiom [12, Theorem 3.1].

Theorem 1.2.18. *Let M be a matroid and $\mathcal{C}$ a signature of its circuits. Then the* *following properties are equivalent:*

1. $\mathcal{C}$ *is a ternary signature.*

2. *For any* $X_1, X_2 \in \mathcal{C}$ *with* $\left(X_1^+ \cap X_2^-\right) \cup \left(X_1^- \cap X_2^+\right) \neq \emptyset$ *and for any* $f \in \left(X_1^+ \backslash X_2^-\right) \cup \left(X_1^- \backslash X_2^+\right)$, *there exists* $X_3 \in \mathcal{C}$ *such that* $f \in X_3 \subseteq (X_1 \cup X_2) \backslash \big((X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)\big)$, *and there exist* $e_1 \in X_1 \cap X_3$ *and* $e_2 \in X_2 \cap X_3$ *such that* $X_1(e_1)X_2(e_2) = X_3(e_1)X_3(e_2)$.

3. *There exists a signature* $\mathcal{C}^*$ *of the cocircuits of* M *such that for any* $X \in \mathcal{C}$ *and any* $Y \in \mathcal{C}^*$ *with* $|X \cap Y| = 2, 3$, *we have* $\big|(X^+ \cap Y^+) \cup (X^- \cap Y^-)\big| \equiv \big|(X^+ \cap Y^-) \cup (X^- \cup Y^+)\big| \mod 3$.

4. $\mathcal{C}$ *has no minor isomorphic to a reorientation of the circuit signature* $\{12, 13, 23\}$ *of* $U_{1,3}$, *or to a reorientation of the circuit signature* $\left\{123, 1\overline{2}4, 134, 23\overline{4}\right\}$ *of* $U_{2,4}$.

1.3 How to Characterize Weak Orientations, Orientations, and Ternary Signatures by Modular Triples

Weak orientations, orientations, and ternary signatures are characterized in the literature in a variety of ways. In Section 1.2, we mentioned characterizations by forbidden minors, circuit elimination, and orthogonality. We provide a new characterization of these circuit signatures, as well as of lifting signatures (which are defined in Section 1.4.1), in terms of modular triples of circuits. These characterizations appear in Theorem 1.3.1. I first discovered this theorem for lifting signatures, a context in which modular triples of circuits naturally appear (the word "naturally" is explained in Section 1.4.1). Later, I realized that the theorem applies to the other signatures as well.

Recall that the *exponent* of a group $\mathfrak{G}$, denoted by $\exp(\mathfrak{G})$, is the smallest positive integer e (if it exists) such that $g^e = 1$ for all $g \in \mathfrak{G}$. If no such integer exists, then $\exp(\mathfrak{G}) = \infty$.

Let $\mathcal{C}$ be a circuit signature of a matroid M. In Theorem 1.3.1, we refer to the following property, which we call the *Well-Distribution Property* (WDP): For each modular triple of signed circuits, (C_1, C_2, C_3), there exist sets I_1, I_2, I_3, and I_4 with $I_1 \uplus I_2 = I_3 \uplus I_4 = I$ so that, up to reorientation,

$$\begin{aligned}
C_1 &= (I \cup I_{13}, I_{12}),\\
C_2 &= \pm(I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}\\
C_3 &= \pm(I_3 \cup I_{23}, I_4 \cup I_{13}).
\end{aligned}$$

(The sets I, I_{13}, I_{12}, and I_{23} are defined in Figure 1.2.1.)

Theorem 1.3.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M.*

1. *$\mathcal{C}$ is a weak orientation of M if and only if the Well-Distribution Property holds.*

2. *$\mathcal{C}$ is an orientation of M if and only if the Well-Distribution Property holds with $I_3 \subseteq I_2$.*

3. *$\mathcal{C}$ is a ternary signature of M if and only if the Well-Distribution Property holds with $I_1 = I_3 = I$.*

4. *Let $\mathfrak{A}$ be an abelian group with $\exp(\mathfrak{A}) > 2$.*

    (a) *Assume M is binary. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if the Well-Distribution Property holds with $I_1 = I_3 = I$.*

(b) *Assume M is not binary. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if* $\exp(\mathfrak{A}) = 3$ *and the Well-Distribution Property holds with* $I_1 = I_3 = I$.

In this section, we prove Theorem 1.3.1(1)–(3); part (4) is proved in Section 1.4.2. When proving Theorem 1.3.1(1)–(3), we use the fact that if $\mathcal{C}$ is an orientation or ternary signature of M, then $\mathcal{C}$ is a weak orientation of M. We can see that orientations are weak orientations by comparing their characterizations by means of the Minty Coloring Property (see Section 1.2.5); and the fact that ternary signatures are weak orientations follows from a comparison of their forbidden minor theorems, Theorems 1.2.18(4) and 1.2.10.

Since orientations and ternary signatures are weak orientations, we can use Theorem 1.3.1(1) to help with the proofs of necessity of parts (2) and (3). For the sake of variety, we use two different techniques to prove the sufficiency of these three results. To prove the sufficiency of part (2), we use Las Vergnas' result that signed circuit elimination for orientations is equivalent to modular signed circuit elimination. To prove the sufficiency of parts (1) and (3), we use forbidden minor arguments. It is possible, however, to prove the sufficiency of all three results using either one of these methods.

The forbidden minor arguments require Lemmas 1.3.2 and 1.3.3.

Lemma 1.3.2. *Let M be a matroid, and let a be an element of M. Let (X_1', X_2') be a modular pair of circuits in M/a. Then the (unique) circuits X_1 and X_2 of M such that $X_1' = X_1 \backslash \{a\}$ and $X_2' = X_2 \backslash \{a\}$ constitute a modular pair of circuits in M.*

Proof. See the proof of Lemma 2.3 in [10]. □

In addition to the WDP, other requirements involving the sets I_1, I_2, and I_3 appear in Theorem 1.3.1(2)–(4). Let Property P be the WDP together with one (or none) of these additional requirements.

Lemma 1.3.3. *Let $\mathcal{C}$ be a circuit signature of a matroid M that satisfies Property P. Then any minor of $\mathcal{C}$ also satisfies Property P.*

Proof. We show that if $\mathcal{C}$ satisfies Property P, then $\mathcal{C}\backslash s$ and $\mathcal{C}/s$ also satisfy Property P. We prove the contrapositive of this statement by induction on $|E(M)|$. The base case, when $|E(M)| = 1$, is vacuously true because M has at most one circuit.

Assume that $|E(M)| > 1$. If $\mathcal{C}\backslash s$ does not satisfy Property P, then $\mathcal{C}\backslash s$ has a modular triple of signed circuits, say (C_1, C_2, C_3), that does not satisfy Property P. But (C_1, C_2, C_3) is also a modular triple of circuits of M, and the signatures of C_i in $\mathcal{C}$ and $\mathcal{C}\backslash s$ are the same. Therefore, $\mathcal{C}$ does not satisfy Property P either.

If $\mathcal{C}/s$ does not satisfy Property P, then $\mathcal{C}/s$ has a modular triple of signed circuits, say (C_1, C_2, C_3), that does not satisfy Property P. Ignoring signatures momentarily, we claim that M has a modular triple of circuits, (D_1, D_2, D_3), such that $C_i \subseteq D_i$. The signature of D_i is an extension of the signature of C_i, so (D_1, D_2, D_3) cannot satisfy Property P because this would imply that (C_1, C_2, C_3) satisfies Property P.

To prove our claim, we apply Lemma 1.3.2 and see that M has circuits D_1, D_2, and D_3 such that $C_i = D_i\backslash\{s\}$ and (D_1, D_2), (D_1, D_3), and (D_2, D_3) are modular pairs of circuits. We know that $D_i = C_i$ or $D_i = C_i \cup \{s\}$. Also, D_1, D_2, and D_3 are distinct circuits because C_1, C_2, and C_3 are distinct. To prove that (D_1, D_2, D_3) is a modular triple of circuits of M, we need only prove that $D_i \subseteq (D_j \cup D_k)$ for distinct i, j, and k. The only way that $D_i \subseteq (D_j \cup D_k)$ can fail is when $D_i = C_i \cup \{s\}$, $D_j = C_j$, and $D_k = C_k$. Suppose this happens. Let E be the ground set of M. Define $H_i^* = \big(E\backslash\{s\}\big)\backslash C_i$. Then (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of $M^*\backslash s$. Also, $\big(H_i^*, H_j^* \cup \{s\}, H_k^* \cup \{s\}\big)$ is a modular triple of copoints of M^*. Since M^* is a single-element extension of $M^*\backslash s$, we can tell from Figure 1.2.2 (see Section 1.2.1) that H_j^* and H_k^* are in the associated modular cut of $M^*\backslash s$, but H_i^* is not in the modular cut.

According to the definition of a modular cut, this is impossible because (H_j^*, H_k^*) is a modular pair and so H_i^* must be in the modular cut as well. This contradiction proves that $D_i \subseteq (D_j \cup D_k)$, which concludes the proof of our claim.

A complete proof of the lemma follows by induction on the number of deletions and contractions that are required to obtain the minor of $\mathcal{C}$. □

Lemmas 1.3.4 and 1.3.5 help prove Theorem 1.3.1(1)–(3). Lemma 1.3.4 allows the freedom to reorient circuit signatures without changing their type, and Lemma 1.3.5 is an easy technical result.

Lemma 1.3.4. *Let $\mathcal{C}$ be a circuit signature of M, and let $A \subseteq E$.*

1. *[1, Proposition 1.7] If $\mathcal{C}$ is a weak orientation of M, then $_{\overline{A}}\mathcal{C}$ is also a weak orientation of M.*
2. *[2, Section 2] If $\mathcal{C}$ is an orientation of M, then $_{\overline{A}}\mathcal{C}$ is also an orientation of M.*
3. *[12, Section 2] If $\mathcal{C}$ is a ternary signature of M, then $_{\overline{A}}\mathcal{C}$ is also a ternary signature of M.*

Lemma 1.3.5. *Let $\mathcal{C}$ be a weak orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. Let i, j, and k be distinct elements of $\{1, 2, 3\}$. If $\{x_1, x_2\} \subseteq (C_i \cap C_j)\backslash C_k$ and $x_1 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$, then $x_2 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$.*

Proof. We may assume that $x_1 \in C_i^+ \cap C_j^+$. Otherwise, we could proceed with the proof using $-C_i$ and $-C_j$. Assume the conclusion is false, and relabel, if necessary, so that $x_2 \in C_i^+ \cap C_j^-$. By Theorem 1.2.9(i), there exists $X_3 \in \mathcal{C}$ with $x_1 \in X_3 \subseteq (C_i \cup C_j)\backslash\{x_2\}$. By Lemma 1.2.2, there is a unique circuit contained in $(C_i \cup C_j)\backslash\{x_2\}$, namely C_k. Thus $X_3 = \pm C_k$, a contradiction because $x_1 \notin C_k$. □

When proving Theorem 1.3.1(1)–(3), we frequently simplify notation by making assumptions about reorientation and negation. These assumptions will be explained in the proof of part (1), but not thereafter.

Proof of Theorem 1.3.1(1). Assume $\mathcal{C}$ is a weak orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Lemma 1.3.4(1), we may assume that

$$C_1 = (I \cup I_{13}, I_{12}).$$

(Technically, we reorient by A so that ${}_{\overline{A}}C_1 = (I \cup I_{13}, I_{12})$. The structure of the signed circuits in the WDP, however, is specified only up to reorientation, so there is no need to complicate the argument with additional notation.)

We show that either $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq C_2^-$. If not, there exist y_1 and y_2, both elements of I_{12}, such that $y_1 \in C_2^+ \cap C_1^-$ and $y_2 \in C_2^- \cap C_1^-$. This contradicts Lemma 1.3.5. We may assume that $I_{12} \subseteq C_2^+$. (Technically, $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq (-C_2)^+$. The structure of C_2 in the WDP, however, is specified only up to negation. By reorientation in I_{23}, we may also assume that $I_{23} \subseteq C_2^-$. (This is possible since I_{23} is disjoint from C_1.) We have found that

$$C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$$

where $I_1 \mathbin{\dot\cup} I_2 = I$.

Notational adjustments to the argument above prove that $I_{13} \subseteq C_3^+$ or $I_{13} \subseteq C_3^-$, and that $I_{23} \subseteq C_3^+$ or $I_{23} \subseteq C_3^-$. Suppose that the elements of $I_{13} \cup I_{23}$ all have the same sign in C_3. We may assume that $I_{13} \cup I_{23} \subseteq C_3^-$. Choose $x \in I_{13}$. When we apply Theorem 1.2.9(ii) to x, C_1, and C_3, we find that $e_1 \in I_{12}$ and $e_2 \in I_{23}$, so $C_1(e_1)C_3(e_2) = (-1)(-1) = +1$. However, by Lemma 1.2.2, $X_4 = \pm C_2$, and in both cases $X_4(e_1)X_4(e_2) = -1$, a contradiction. Thus,

$$C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$$

where $I_3 \uplus I_4 = I$.

Now assume that $\mathcal{C}$ satisfies the WDP. By Lemma 1.3.3, any minor of $\mathcal{C}$ must also satisfy the WDP. Thus an induced $U_{1,3}$ circuit signature must be isomorphic to a reorientation of $\{12, 13, 2\overline{3}\}$. Using Theorem 1.2.10, we conclude that $\mathcal{C}$ is a weak orientation of M. □

Proof of Theorem 1.3.1(2). Assume that $\mathcal{C}$ is an orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Theorem 1.3.1(1) and Lemma 1.3.4(2) (and possibly replacing C_2 or C_3 with their negatives), we may assume that

$$\begin{aligned} C_1 &= (I \cup I_{13}, I_{12}), \\ C_2 &= (I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and} \\ C_3 &= (I_3 \cup I_{23}, I_4 \cup I_{13}), \end{aligned}$$

where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$.

Choose $y \in I_{12}$. By applying signed circuit elimination to y, C_1, and C_2, we find that $\mathcal{C}$ has a signed circuit $C \subseteq (C_1 \cup C_2)\backslash\{y\}$ with

$$C^+ \subseteq I \cup I_{13} \cup I_{12}\backslash\{y\} \text{ and } C^- \subseteq I_2 \cup I_{12}\backslash\{y\} \cup I_{23}.$$

By Lemma 1.2.2, $C = \pm C_3$. Thus $(I_3 \cup I_{23}) \subseteq (I_2 \cup I_{23})$, which implies that $I_3 \subseteq I_2$.

Now assume that $\mathcal{C}$ satisfies the WDP with $I_3 \subseteq I_2$. To prove that $\mathcal{C}$ is an orientation of M, we need to prove the circuit elimination axiom for orientations. This axiom appears in Section 1.2.4. In that section, we also mention that circuit elimination only needs to be verified for modular pairs of signed circuits. Let (C_1, C_2) be a modular pair of signed circuits such that $C_1 \neq \pm C_2$, and let $e \in C_1^+ \cap C_2^-$. By Lemma 1.2.2, there exists a unique circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2)\backslash\{e\}$. Moreover, (C_1, C_2, C_3) is a modular triple. According to our notation, we know that $e \in I_{12}$.

We must prove that for $\tau = +$ or $\tau = -$, $(\tau C_3)^+ \subseteq (C_1^+ \cup C_2^+)\backslash\{e\}$ and $(\tau C_3)^- \subseteq (C_1^- \cup C_2^-)\backslash\{e\}$. Up to reorientation, we know that

$$C_1 = (I \cup I_{13}, I_{12}),$$
$$C_2 = \pm(I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$
$$C_3 = \pm(I_3 \cup I_{23}, I_4 \cup I_{13}),$$

where $I_1 \mathbin{\dot\cup} I_2 = I_3 \mathbin{\dot\cup} I_4 = I$ and $I_3 \subseteq I_2$. By construction, e has apposite signs in C_1 and C_2 (both before and after reorientation). Thus $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$. Since we assumed that $I_3 \subseteq I_2$, it follows that

$$(\tau C_3)^+ \subseteq C_1^+ \cup C_2^+ \text{ and } (\tau C_3)^- \subseteq C_1^- \cup C_2^-$$

for $\tau = +$ or $\tau = -$. □

Proof of Theorem 1.3.1(3). Assume that $\mathcal{C}$ is a ternary signature, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Theorem 1.3.1(1) and Lemma 1.3.4(3) (and possibly replacing C_2 or C_3 with their negatives), we may assume that

$$C_1 = (I \cup I_{13}, I_{12}),$$
$$C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$
$$C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13}),$$

where $I_1 \cupdot I_2 = I_3 \cupdot I_4 = I$.

Choose $x \in I_{13}$, and let C_1, C_2, and x play the respective roles of X_1, X_2, and f in Theorem 1.2.18(2). Thus there exists $X_3 \in \mathcal{C}$ such that $X_3 \subseteq (C_1 \cup C_2) \backslash (I_2 \cup I_{12})$. But C_3 and $-C_3$ are the only signed circuits contained in $(C_1 \cup C_2) \backslash I_{12}$, so $X_3 = \pm C_3$. However, $I \subseteq C_3$, so $I_2 = \emptyset$.

An identical argument shows that $I_4 = \emptyset$, which concludes the proof of necessity.

To prove sufficiency, assume that $\mathcal{C}$ satisfies the WDP with $I_1 = I_3 = I$. By Theorems 1.3.1(1) and 1.2.10, $\mathcal{C}$ has no minor isomorphic to a reorientation of $\{12, 13, 23\}$.

According to Lemma 1.3.3, any minor that is a signature of $U_{2,4}$ must also satisfy the WDP with $I_1 = I_3 = I$. We claim that such a minor is a reorientation of $\left\{12\overline{3}, 1\overline{2}4, 13\overline{4}, 234\right\}$. There is no way to reorient this signature so that exactly two circuits have positive signatures, thus $\mathcal{C}$ has no minor isomorphic to a reorientation of the signatures in Theorem 1.2.18(4). It follows that $\mathcal{C}$ is a ternary signature of M.

We conclude with a proof of our claim. Since any three circuits of $U_{2,4}$ form a modular triple, we know that the signatures of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 3, 4\}$ are some reorientation of $12\overline{3}$, $1\overline{2}4$, and $13\overline{4}$. We also know that the signatures of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$ are some reorientation of $12\overline{3}$, $\overline{1}24$, and $23\overline{4}$. Putting these two facts together, we see that the circuit signature of $U_{2,4}$ must be some reorientation of $\left\{12\overline{3}, 1\overline{2}4, 13\overline{4}, 234\right\}$. $\square$

1.4 Lifting Signatures

1.4.1 Definitions

Now we generalize Section 1.2.2, where gains enabled the construction of graphic-matroid lifts. The main idea is to replace information obtained from graphs with information obtained from matroid circuit signatures.

Let $\Phi = (\Gamma, \phi)$ be a gain graph with gain group $\mathfrak{G}$. We can think of Γ as a directed graph because ϕ oriented the edges in order to assign gains. There is a standard way of associating this directed graph with an orientation $\mathcal{C}$ of the graphic matroid $G(\Gamma)$ (see [4, Section 1.1]). Arbitrarily assign an orientation to each circle of Γ; an element of a signed circuit is positive if its direction agrees with the orientation assigned that circle, and it is negative otherwise.

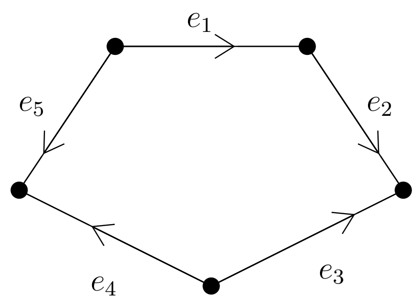


Figure 1.4.1: This is a circle of Φ. The arrows on the edges indicate the orientations prescribed by ϕ.

Suppose the circle B in Figure 1.4.1 is in Γ. According to Section 1.2.2, B is balanced if and only if

$$\phi(e_1)\phi(e_2)\phi(e_3)^{-1}\phi(e_4)\phi(e_5)^{-1} = 1.$$

Balance can also be defined using the circuit signature $\mathcal{C}$ we described above. In our example,

$$\big(\{e_1, e_2, e_4\}, \{e_3, e_5\}\big) \in \mathcal{C}.$$

Assuming that the gain group is abelian, B is balanced if and only if

$$\prod_{e \in B^+} \phi(e) \prod_{e \in B^-} \phi(e)^{-1} = 1.$$

We require that the gain group be abelian; otherwise, this product may not be well defined.

Our example illustrates how the circuit signature that is associated with a gain graph provides a different way of determining which circles are balanced. Moreover, this method lends itself to a matroid generalization, where circuit signatures are used to determine whether or not a matroid circuit is balanced. Now we give the formal definitions that are necessary for this generalization.

Let M be a matroid on E, let $\mathcal{C}$ be a circuit signature of M, and let $\mathfrak{A}$ be an abelian group. A *gain mapping* ϕ is a function from E into $\mathfrak{A}$. We call $\mathfrak{A}$ the *gain group*.

Let C be a circuit of M, so C is the support of two signed circuits in $\mathcal{C}$. Suppose one of these signed circuits is $\big(\{a_1,\ldots,a_p\},\{b_1,\ldots,b_n\}\big)$. We define the *gain* of C to be

$$\phi(C) = \prod_{a\in C^+} \phi(a) \prod_{b\in C^-} \phi(b)^{-1}.$$

We say that C is *balanced* if $\phi(C) = 1$. Whether or not C is balanced is independent of which of the two signed circuits with support C is used for the computation. Let $\mathcal{B}(\phi,\mathcal{C})$ denote the class of balanced circuits. If $\mathcal{C}$ is clear from context, we write $\mathcal{B}(\phi)$. If $\mathcal{B}(\phi)$ is a linear class of circuits, we can apply Dowling and Kelly's lift construction (as in Section 1.2.1) to obtain $L\big(M,\mathcal{B}(\phi)\big)$, an elementary lift of M.

It is certainly not the case that $\mathcal{B}(\phi,\mathcal{C})$ is linear for all choices of ϕ and $\mathcal{C}$. For example, consider the matroid $U_{2,4}$ with orientation

$$\big\{1\bar{2}3, 1\bar{2}4, 1\bar{3}4, 2\bar{3}4\big\}. \tag{1.4.1}$$

Let the gain group be $\mathbb{R}^+$, and define ϕ by

$$\phi(1) = 1, \quad \phi(2) = 0, \quad \text{and} \quad \phi(3) = \phi(4) = -1.$$

Then $\phi(123) = \phi(124) = 0$, but $\phi(134) = 1$. Thus $\mathcal{B}(\phi, \mathcal{C})$ is not linear, and the lift construction cannot be applied. There certainly exists a gain mapping τ that makes $\mathcal{B}(\tau, \mathcal{C})$ linear. However, we want to generalize Section 1.2.2 (the graphical case), where $\mathcal{B}(\phi)$ is always linear. Thus the $U_{2,4}$ example teaches us that we must be more selective when choosing the circuit signature.

The discussion above inspires the following definitions. A matroid M can be *lifted by gains in* $\mathfrak{A}$ if M has a circuit signature $\mathcal{C}$ such that $\mathcal{B}(\phi, \mathcal{C})$ is linear for all $\phi : E \to \mathfrak{A}$. In this case, we call $\mathcal{C}$ a *lifting signature for gains in* $\mathfrak{A}$. For example, the orientation associated with a graph Γ is a lifting signature of the graphic matroid $G(\Gamma)$ (for gains in any group).

Now we ask, which matroids (in addition to graphic matroids) have lifting signatures? Since linear classes of circuits are central to the definition of a lifting signature and since they are defined in terms of modular triples, it is natural that modular triples be used to characterize lifting signatures.

1.4.2 How to Characterize Lifting Signatures by Modular Triples

In this section we prove Theorem 1.3.1(4), which characterizes lifting signatures in terms of modular triples of signed circuits. The following Theorem is then an immediate consequence of Theorem 1.3.1.

Theorem 1.4.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M, and let $\mathfrak{A}$ be an abelian group with* $\exp(\mathfrak{A}) > 2$*. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if $\mathcal{C}$ is a ternary signature and* $\exp(\mathfrak{A}) = 3$ *when M is not binary.*

Before proving Theorem 1.3.1(4), we prove two lemmas, which parallel the reorientation and technical lemmas that contributed to the proofs of Theorem 1.3.1(1)–(3).

Given a gain mapping ϕ, define a new gain mapping ϕ_A by

$$\phi_A(e) = \begin{cases} \phi(e) & \text{if } e \notin A, \\ \phi(e)^{-1} & \text{if } e \in A. \end{cases}$$

Lemma 1.4.2. *Let $\mathcal{C}$ be a circuit signature of M, let $A \subseteq E$, and let $\mathfrak{A}$ be an abelian group.*

1. *For each gain mapping ϕ, $\mathcal{B}(\phi_A, {}_{\overline{A}}\mathcal{C}) = \mathcal{B}(\phi, \mathcal{C})$.*
2. *$\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if ${}_{\overline{A}}\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$.*

Proof. Let ϕ be a gain mapping, and let C be a circuit of M. Throughout this proof, $\phi(C)$ is calculated using $\mathcal{C}$ and $\phi_A(C)$ is calculated using ${}_{\overline{A}}\mathcal{C}$.

Assume that C is the support of the signed circuit

$$\big(\{p_1, \dots, p_r, a_1, \dots, a_s\}, \{n_1, \dots, n_t, b_1, \dots, b_q\}\big)$$

of $\mathcal{C}$, where $A \cap C = \{a_1, \dots, a_s, b_1, \dots, b_q\}$. Then C is the support of the signed circuit

$$\big(\{p_1, \dots, p_r, b_1, \dots, b_q\}, \{n_1, \dots, n_t, a_1, \dots, a_s\}\big)$$

of ${}_{\overline{A}}\mathcal{C}$. Accordingly,

$$\begin{aligned} \phi_A(C) &= \prod_{i=1}^{r} \phi_A(p_i) \prod_{i=1}^{q} \phi_A(b_i) \prod_{i=1}^{t} \phi_A(n_i)^{-1} \prod_{i=1}^{s} \phi_A(a_i)^{-1} \\ &= \prod_{i=1}^{r} \phi(p_i) \prod_{i=1}^{s} \phi(a_i) \prod_{i=1}^{t} \phi(n_i)^{-1} \prod_{i=1}^{q} \phi(b_i)^{-1} \\ &= \phi(C). \end{aligned}$$

It follows immediately that $\mathcal{B}(\phi_A, {}_{\overline{A}}\mathcal{C}) = \mathcal{B}(\phi, \mathcal{C})$.

To prove part (2), assume that $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. So there exist a gain mapping ϕ and a modular triple of circuits, (C_1, C_2, C_3), such that $\phi(C_1) = \phi(C_2) = 1$ and $\phi(C_3) \neq 1$. From part (1), it follows that $\phi_A(C_1) = \phi_A(C_2) = 1$ and $\phi_A(C_3) \neq 1$. Thus ${}_{\overline{A}}\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$.

Now assume that ${}_{\overline{A}}\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. We just proved that this implies that ${}_{\overline{A}}({}_{\overline{A}}\mathcal{C})$ is not a lifting signature for gains in $\mathfrak{A}$. But ${}_{\overline{A}}({}_{\overline{A}}\mathcal{C}) = \mathcal{C}$. $\square$

Lemma 1.4.3. *Let $\mathfrak{A}$ be an abelian group with $\exp(\mathfrak{A}) > 2$, let $\mathcal{C}$ be a lifting signature of M for gains in $\mathfrak{A}$, and let (C_1, C_2, C_3) be a modular triple of signed circuits. Let i, j, and k be distinct element of $\{1, 2, 3\}$.*

1. *Assume $\{x_1, x_2\} \subseteq (C_i \cap C_j) \backslash C_k$. If $x_1 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$, then $x_2 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$.*
2. *Assume x, y, and z are each in exactly two of C_i, C_j, and C_k. If $x \in C_i^+ \cap C_k^-$, $y \in C_j^+ \cap C_i^-$, $z \in C_j^-$, and $z \in C_k$, then $z \in C_k^+$.*
3. *Assume $y \in (C_i \cap C_j) \backslash C_k$ and $w \in C_i \cap C_j \cap C_k$. If $y \in C_i^- \cap C_j^+$ and $w \in C_i^+$, then $w \in C_j^+$.*

Proof. Throughout this proof, let $g \in \mathfrak{A}$ have order greater than 2.

For part (1), we may assume that $x_1 \in (C_i^+ \cap C_j^+)$. Otherwise, we could proceed with the proof using the modular triple $(-C_i, -C_j, C_k)$. Suppose the conclusion is false. By relabeling if necessary, we can assume that $x_2 \in (C_i^+ \cap C_j^-)$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{x_1, x_2\}, \\ 1 & \text{otherwise.} n \end{cases}$$

Then $\phi(C_j) = g \cdot g^{-1} = 1$ and $\phi(C_k) = 1$, but $\phi(C_i) = g \cdot g = g^2 \neq 1$, which contradicts the assumption that $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$.

For part (2), suppose $z \in C_k^-$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{x, y, z\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_i) = \phi(C_j) = g \cdot g^{-1} = 1$, but $\phi(C_k) = g^{-1} \cdot g^{-1} = (g^{-1})^2 \neq 1$, a contradiction.

For part (3), suppose $w \in C_j^-$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{w, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_i) = \phi(C_j) = g \cdot g^{-1} = 1$. It is not known whether w is positive or negative in C_k, so $\phi(C_k) = g$ or $\phi(C_k) = g^{-1}$. In either case, $\phi(C_k) \neq 1$, a contradiction. □

Theorem 1.3.1(4) is divided into two parts, depending on whether or not the matroid is binary. The two parts are made necessary by Theorem 1.2.1, which states that in a binary matroid, the intersection of the three circuits in a modular triple is empty, but that a nonbinary matroid has a modular triple for which this is not the case.

Proof of Theorem 1.3.1(4). Assume $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Lemma 1.4.2(2) (reorientation), we may assume that

$$C_1 = (I \cup I_{13}, I_{12}).$$

We will show that either $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq C_2^-$. If not, there exist y_1 and y_2, both elements of I_{12}, such that $y_1 \in C_1^- \cap C_2^-$ and $y_2 \in C_1^- \cap C_2^+$. This contradicts Lemma 1.4.3(1). We may assume that $I_{12} \subseteq C_2^+$ and that $I_{23} \subseteq C_2^-$. Applying Lemma

1.4.3(3), the elements of I have the same sign in C_2 as the elements of I_{12}. Thus

$$C_2 = (I \cup I_{12}, I_{23}).$$

An argument similar to the one above proves that either $I_{13} \subseteq C_3^+$ or $I_{13} \subseteq C_3^-$. Furthermore, by Lemmas 1.4.3(2) and 1.4.3(3), we find that the elements of $I \cup I_{23}$ and those of I_{13} have opposite signs in C_3. (Use 1.4.3(2) for I_{23} and 1.4.3(3) for I.) Thus

$$C_3 = (I \cup I_{23}, I_{13}).$$

We have proved the necessity of part (4a).

To prove the necessity of part (4b), we must show that $\exp(\mathfrak{A}) = 3$. Suppose $\exp(\mathfrak{A}) \neq 3$, so there exists $g \in \mathfrak{A}$ such that $g^3 \neq 1$. Since M is not binary, we apply Lemma 1.2.1 to find a modular triple of signed circuits, (C_1, C_2, C_3), with nonempty intersection. We must show that $\mathcal{C}$ is not a lifting signature. By the above argument, we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = \pm(I \cup I_{12}, I_{23})$, and $C_3 = \pm(I \cup I_{23}, I_{13})$. Choose $w \in I$, $x \in I_{13}$, and $z \in I_{23}$, and define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e = x, \\ g^{-1} & \text{if } e = w \text{ or } z, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_1) = \phi(C_2) = 1$, but $\phi(C_3)$ is $(g^{-1})^3$ or g^3, neither of which is 1. Thus $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. This contradicts our hypothesis, so $\exp(\mathfrak{A}) = 3$.

Now we prove sufficiency. Let (C_1, C_2, C_3) be a modular triple of signed circuits. We must prove that $\mathcal{C}$ is a lifting signature. By reorientation (and possibly replacing one or both of C_2 and C_3 by their negatives), we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I \cup I_{12}, I_{23})$, and $C_3 = (I \cup I_{23}, I_{13})$.

Let ϕ be a gain mapping. We must show that if $\phi(C_1) = \phi(C_2) = 1$, then $\phi(C_3) = 1$. (The other two combinations follow by relabeling.) If

$$\phi(C_1) = \phi(C_2) = 1,$$

then

$$\prod_{w \in I} \phi(w) \prod_{x \in I_{13}} \phi(x) \prod_{y \in I_{12}} \phi(y)^{-1} = \prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y) \prod_{z \in I_{23}} \phi(z)^{-1} = 1.$$

Thus

$$\begin{aligned} \phi(C_3) &= \prod_{w \in I} \phi(w) \prod_{z \in I_{23}} \phi(z) \prod_{x \in I_{13}} \phi(x)^{-1} \\ &= \left(\prod_{w \in I} (\phi(w) \right) \left(\prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y) \right) \left(\prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y)^{-1} \right) \\ &= \prod_{w \in I} (\phi(w))^3. \end{aligned}$$

If M is binary, then $I = \emptyset$ (see Lemma 1.2.1). If M is not binary, then $\exp(\mathfrak{A}) = 3$. In both cases, $\phi(C_3) = 1$. □

1.4.3 Using Gains to Lift Binary and Ternary Matroids

So far, all results about lifting signatures mandate that the gain group have exponent greater than 2. The root cause of this is Lemma 1.4.3, whose proof requires an element of order greater than 2. Gain groups of exponent 2 have a different effect on lifting signatures because $\phi(e) = \phi(e)^{-1}$ for all elements of the matroid. Thus all circuit signatures behave like the all-positive signature.

Theorem 1.4.4 classifies the matroids that can be lifted by gains from a group of exponent greater than 2. Theorem 1.4.5 is a classification for gain groups of exponent 2.

Theorem 1.4.4. *Let M be a matroid, and let $\mathfrak{A}$ be an abelian group such that* $\exp(\mathfrak{A}) > 2$. *Then M can be lifted by gains in $\mathfrak{A}$ if and only if M is ternary and* $\exp(\mathfrak{A}) = 3$ *when M is not binary. Moreover, the lifting signature is the ternary signature associated with M, which is unique up to reorientation.*

Proof. M can be lifted by gains in $\mathfrak{A}$ if and only if M has a lifting signature for gains in $\mathfrak{A}$, call it $\mathcal{C}$. From Theorem 1.4.1, we see that $\mathcal{C}$ is also a ternary signature. But M has a ternary signature if and only if M is ternary. Moreover, Theorem 1.2.17 guarantees that a ternary matroid has precisely one ternary signature, up to reorientation. □

Theorem 1.4.5. *Let $\mathfrak{A}$ be an abelian group with* $\exp(\mathfrak{A}) = 2$, *and let $\mathcal{C}$ be a circuit signature of a matroid M. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if M is binary.*

Proof. Assume that M is not binary. By Lemma 1.2.1, there exists a modular triple of signed circuits, (C_1, C_2, C_3), with nonempty intersection. Let $w \in I$ and $y \in I_{12}$, and let $g \in \mathfrak{A}$ be any element other than 1. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{w, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_1) = \phi(C_2) = 1$, but $\phi(C_3) = g \neq 1$. Thus $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$.

Now assume that M is binary. Let (C_1, C_2, C_3) be a modular triple of signed circuits, and let ϕ be a gain mapping. By Lemma 1.2.1, we know that $I = \emptyset$. Since $\exp(\mathfrak{A}) = 2$, $\phi(e) = \phi(e)^{-1}$ for all $e \in E$, so we may assume that $\mathcal{C}$ is the all-positive signature. For distinct i, j, and k, we must show that if $\phi(C_i) = \phi(C_j) = 1$, then $\phi(C_k) = 1$ as well. We show this for one case; the other cases are identical up to renaming sets. Assume

$$\phi(C_1) = \phi(C_2) = 1.$$

Thus

$$\prod_{x \in I_{13}} \phi(x) \prod_{y \in I_{12}} \phi(y) = \prod_{y \in I_{12}} \phi(y) \prod_{z \in I_{23}} \phi(z) = 1.$$

Then

$$\phi(C_3) = \prod_{x \in I_{13}} \phi(x) \prod_{x \in I_{23}} \phi(z) = \prod_{y \in I_{12}} (\phi(y))^2 = 1.$$

□

1.5 Applications

This section consists of a variety of applications of Theorem 1.3.1. First, we provide quick, easy proofs of Corollary 1.5.1, which is largely a collection of known facts about orientations, weak orientations, and ternary signatures. Following our proof of each part, we give a reference for a previously known alternative proof. Next, we prove that the Fano matroid is not orientable. This fact follows from Corollary 1.5.1(2). However, we apply Theorem 1.3.1(2) directly in order to give the reader a better understanding of this theorem. Lastly, we go through the mechanics of using gains to construct a lift of a particular matroid.

Corollary 1.5.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M, and let $\mathfrak{A}$ be an abelian group where* $\exp(\mathfrak{A}) > 2$*, and* $\exp(\mathfrak{A}) = 3$ *if M is not binary.*

1. *Assume M is binary. The following are equivalent: $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$, $\mathcal{C}$ is an orientation, $\mathcal{C}$ is a weak orientation, and $\mathcal{C}$ is a ternary signature.*

2. A binary matroid is orientable if and only if it is regular.

3. Assume M is regular and $\mathcal{C}$ is an orientation. Then, up to reorientation, $\mathcal{C}$ is unique.

4. If M is not binary and $\mathcal{C}$ is a ternary signature, then $\mathcal{C}$ is a weak orientation but is not an orientation.

Proof. (1) By Lemma 1.2.1, if (C_1, C_2, C_3) is a modular triple of circuits of M, then $C_1 \cap C_2 \cap C_3 = \emptyset$. The proof now follows immediately from Theorem 1.3.1. (Previous proof: Combine Theorems 1.2.6, 1.2.7, 1.2.18(4), and 1.2.10. Of course, the inclusion of lifting signatures in this result is new.)

(2) A binary matroid M is orientable if and only if it has an orientation $\mathcal{C}$. By part (1), this is equivalent to $\mathcal{C}$ being a ternary signature. Hence M is ternary and binary, and therefore M is regular. (Previous proof: See [1, Proposition 7.9.3].)

(3) M is both binary and ternary. Since it is binary, part (1) says that $\mathcal{C}$ is also a ternary signature. But by Theorem 1.2.17, $\mathcal{C}$ is unique up to reorientation. (Previous proof: See [1, Corollary 7.9.4].)

(4) Since M is not binary, Lemma 1.2.1 guarantees the existence of a modular triple of circuits, (C_1, C_2, C_3), such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$. Thus I in Theorem 1.3.1 is nonempty. Since $\mathcal{C}$ is ternary, Theorem 1.3.1 indicates that, up to reorientation and negation,

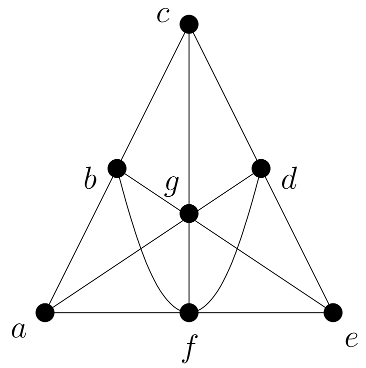
$$\begin{aligned} C_1 &= (I \cup I_{13}, I_{12}), \\ C_2 &= (I \cup I_{12}, I_{23}), \text{ and} \\ C_3 &= (I \cup I_{23}, I_{13}). \end{aligned}$$

If $\mathcal{C}$ is also an orientation, then, up to reorientation and negation,

$$C_1 = (I \cup I_{13}, I_{12}),$$
$$C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$
$$C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$$

for some $I_1 \uplus I_2 = I_3 \uplus I_4 = I$ with $I_3 \subseteq I_2$. Thus $I_1 = I_3 = I$, and $I_2 = I_4 = \emptyset$. This contradicts $I_3 \subseteq I_2$. The result follows because ternary signatures are weak orientations. (Previous proof: See [12, Theorem 4.3].) □

Figure 1.5.1: A geometric representation of F_7.



Theorem 1.5.2. *The Fano matroid $F_7 = PG(2,2)$ is not orientable.*

Proof. Any two three-point circuits, C_1 and C_2, together with $C_1 \triangle C_2$, are a modular triple of circuits.

Assume $\mathcal{C}$ is an orientation of F_7. We show that the WDP of Theorem 1.3.1(2) cannot be satisfied. Since F_7 is binary, $I = \emptyset$. We may assume that $\mathcal{C}$ contains the signed circuits

$$abc, \quad cfg, \quad \text{and} \quad cde.$$

Since $\big(\{a,b,c\}, \{c,f,g\}, \{a,b,f,g\}\big)$ is a modular triple of circuits, the WDP guarantees that

$$ab\overline{fg}$$

is also an element of $\mathcal{C}$. Similarly, $\mathcal{C}$ contains the signed circuits

$$ab\overline{de} \text{ and } de\overline{fg}.$$

Since $\big(\{a, e, f\}, \{b, d, f\}, \{a, b, d, e\}\big)$ is a modular triple of circuits, their signatures in $\mathcal{C}$ must be some reorientation of $ae\overline{f}$, $\overline{bd}f$, and $a\overline{bd}e$. But we already know $ab\overline{de}$ is a signed circuit in $\mathcal{C}$, so we must reorient this modular triple by b and e, and possibly by f. Thus $\mathcal{C}$ contains

$$\text{either } \big(a\overline{ef} \text{ and } b\overline{d}f\big) \text{ or } \big(a\overline{e}f \text{ and } b\overline{df}\big). \tag{1.5.1}$$

Using similar techniques, we find that $\mathcal{C}$ contains

$$\text{either } \big(a\overline{dg} \text{ and } b\overline{e}g\big) \text{ or } \big(a\overline{d}g \text{ and } b\overline{eg}\big). \tag{1.5.2}$$

But these techniques also show that $\mathcal{C}$ contains

$$\text{either } \big(a\overline{dg} \text{ and } bd\overline{f}\big) \text{ or } \big(ad\overline{g} \text{ and } b\overline{df}\big).$$

The first possibility contradicts (1.5.1) and the second contradicts (1.5.2). □

As a final application, we use gains in $\mathbb{Z}_3^+$ to show that the matroid of the graph in Figure 1.5.2 is an elementary lift of $U_{2,4}$. According to Theorem 1.4.4, the lifting signature is the ternary signature of $U_{2,4}$, which is $\big\{\overline{12}3, \overline{1}24, 134, \overline{23}4\big\}$. (We found this signature in Section 1.2.6).

Consider the gain mapping ϕ, where $\phi(2) = \phi(3) = \phi(4) = 1$ and $\phi(1) = 0$. From

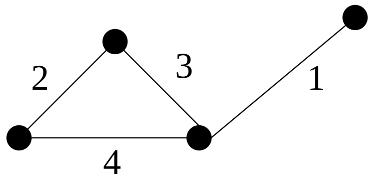


Figure 1.5.2: The matroid of this graph is a lift of $U_{2,4}$ that can be constructed using gains.

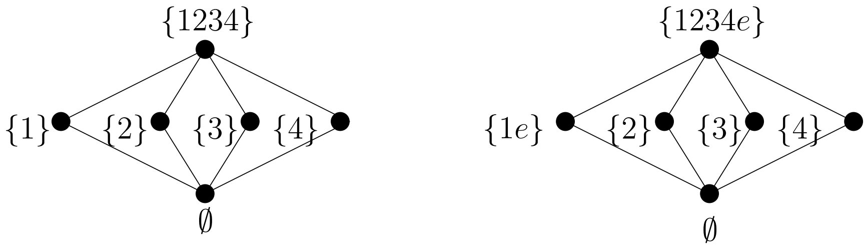


Figure 1.5.3: These are the lattices of $U_{2,4}$ (left) and one if its single-element extensions (right).

Section 1.2.1, we know that

$$L(U_{2,4}, \mathcal{B}(\phi)) = \big((U_{2,4} +_{\mathcal{M}} e)/e\big)^*$$

is a lift of $U_{2,4}$ where $\mathcal{M}$ is determined by (1.2.1). Since $\mathcal{B}(\phi) = \{234\}$ and $\mathcal{B}(\phi)^* = \{1\}$, $\mathcal{M} = \{1, 1234\}$. In Figure 1.5.3, we show the lattices of flats of both $U_{2,4}$ and $U_{2,4}+_{\mathcal{M}}e$. It is easy to see that $(U_{2,4}+_{\mathcal{M}}e)/e$, the dual of $L\big(U_{2,4}, \mathcal{B}(\phi)\big)$, is the matroid of the graph shown in Figure 1.5.4. It follows that $L\big(U_{2,4}, \mathcal{B}(\phi)\big)$ is the matroid of the graph in Figure 1.5.2.

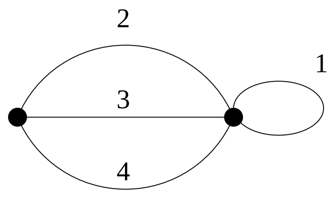


Figure 1.5.4: The matroid $(U_{2,4} +_{\mathcal{M}} e)/e$ is the graphic matroid of this graph.

1.6 Questions

We now know that some of the elementary lifts of a ternary matroid M can be constructed using gains in an abelian group $\mathfrak{A}$ (where $\mathfrak{A}$ has certain restrictions depending on the matroid). How many of the lift matroids of M can be constructed in this way? In particular, for which matroids can all of their lifts be constructed using gains?

Orientations, weak orientations, and ternary signatures are the only matroid circuit signatures that I have found in the literature. Possibly, our modular triple characterizations can help us find other interesting matroid circuit signatures. For example, what if the WDP holds with $I_1 \subseteq I_4$? Do we get a circuit signature that can be characterized by either a signed circuit elimination axiom or a list of forbidden minors?

Let Φ be a gain graph. In [18, Theorem 4.1], Zaslavsky shows that $L(\Phi)$ is representable over any field that contains the gain group as an additive subgroup. For a ternary matroid M, is $L\big(M, \mathcal{B}(\phi)\big)$ representable over some field? (Of course, $\mathcal{B}(\phi)$ is calculated using the ternary signature.)

In the case where M is a graphic matroid, Slilaty showed that $L\big(M, \mathcal{B}(\phi)\big)$ is orientable [13]. For an arbitrary ternary matroid M, is $L\big(M, \mathcal{B}(\phi)\big)$ orientable? Perhaps Theorem 1.3.1(2) can help answer this question.

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Chapter 2

Biased Expansions of Biased Graphs

2.1 Introduction

Given a geometric lattice L of rank r, there is an associated monic polynomial of degree r called the *characteristic polynomial* $p_{L}(\lambda)$. If L is the lattice of flats of a graphic matroid, then $p_{L}(\lambda)$ is closely related to the chromatic polynomial of the associated graph [6, Proposition 7.5.1]. If L is the intersection lattice of a finite collection of hyperplanes in $\mathbb{R}^n$, then $\left|p_{L}(-1)\right|$ is the number of regions formed by the hyperplanes [5, Section 2, Theorem A].

A *hyperplane arrangement* is a finite collection of hyperplanes in an n-dimensional vector space. We allow the *degenerate hyperplane*, which is the entire vector space. If two hyperplane arrangements are closely related, we wonder how their characteristic polynomials are related. In [1], Ehrenborg and Readdy answered this question for a *frame arrangement* $\mathcal{A}$ and its *Dowlingization* $D_m(\mathcal{A})$. For frame arrangements in $\mathbb{R}^n$

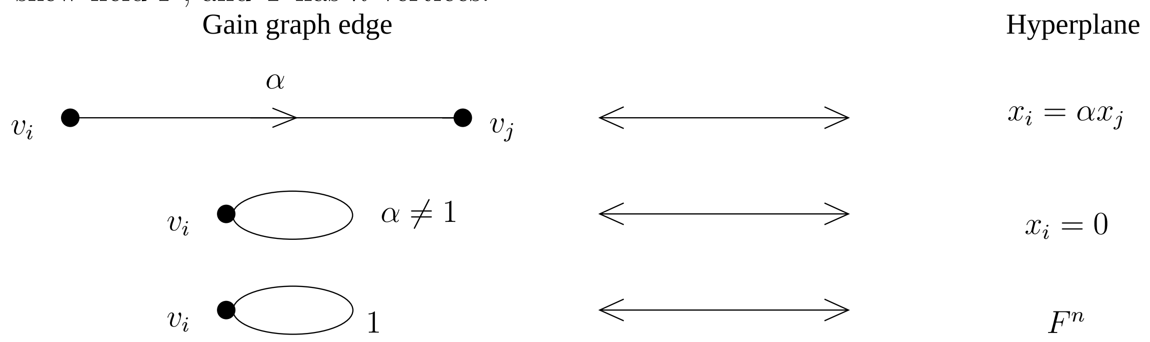
with positive real coefficients, they showed that

$$p_{(D_m(\mathcal{A}))^\bullet}(\lambda) = m^n p_{\mathcal{A}^\bullet}\left(\frac{\lambda - 1}{m} + 1\right), \tag{2.1.1}$$

where $\mathcal{A}^\bullet$ is the hyperplane arrangement consisting of the members of $\mathcal{A}$ together with the coordinate hyperplanes not already in $\mathcal{A}$.

A *frame arrangement* is a hyperplane arrangement consisting of hyperplanes of the form $x_i = \alpha x_j$ ($i \neq j$, $\alpha \neq 0$) or $x_i = 0$. Let $\mathcal{A}$ be a positive real frame arrangement, and let ζ be a primitive mth root of unity. The *Dowlingization* $D_m(\mathcal{A})$ consists of the complex hyperplanes $x_i = \zeta^h \alpha x_j$, $0 \leq h \leq m-1$, together with any coordinate hyperplanes in $\mathcal{A}$. Ehrenborg and Readdy mention that both arrangements can be encoded as gain graphs (see Figure 2.1.1). However, they do not employ the theory of gain and biased graphs in their proofs.

Figure 2.1.1: This is a description of how a gain graph Φ encodes the hyperplanes of a frame arrangement $\mathcal{H}(\Phi)$. The gains are elements of a multiplicative subgroup of a skew field F, and Φ has n vertices.



Armed with the theory of gain and biased graphs, we extend (2.1.1). First we translate this equation to one about gain graphs. Let the gain graph Φ encode $\mathcal{A}$, and let $\langle\zeta\rangle\Phi$ encode $D_m(\mathcal{A})$. (Here $\langle\zeta\rangle\Phi$ is just the name of a gain graph, but I chose this name carefully.) We explain in Section 2.2 (see (2.2.3)) that the characteristic polynomial of a frame arrangement equals the *chromatic polynomial* $\chi_\Phi(\lambda)$ of the associated gain graph Φ. Thus we can rewrite (2.1.1) as

$$\chi_{(\langle\zeta\rangle\Phi)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda - 1}{m} + 1\right). \tag{2.1.2}$$

Let $\mathfrak{G}$ be a group with m elements, and let Δ be an ordinary (not biased) graph. Zaslavsky proved results in [9] that, when combined, yield the following formula:

$$\chi_{(\mathfrak{G}\Delta)^\bullet}(\lambda) = m^n \chi_{\langle\Delta\rangle^\bullet}\left(\frac{\lambda - 1}{m} + 1\right). \tag{2.1.3}$$

The resemblance between (2.1.2) and (2.1.3) is not coincidental.

Corollary 2.3.1, a consequence of our main result, contains both (2.1.2) and (2.1.3) as special cases. The corollary includes a formula like those above for a *group expansion of a gain graph*, a concept we introduce. Equation (2.1.2) is the case in which a gain graph with positive real gains is expanded by the group generated by a primitive root of unity. In (2.1.3), a balanced biased graph is expanded by a finite group.

The main result in this chapter is Theorem 2.3.3, a biased-graph generalization of Corollary 2.3.1. This theorem is about the chromatic polynomial of a *biased expansion of a biased graph*. This new type of biased graph is an abstraction of a group expansion of a gain graph. The inspiration for this level of generality is Zaslavsky's work on biased expansions of ordinary graphs (first defined in [9, Example 3.8], with significant development in [13] and [12]).

We employ several aspects of biased graph theory to prove Theorem 2.3.3. For certain expansions of gain graphs, we use gain-graph coloring, a concept introduced in [9, Section 4]. The general proof incorporates algebraic results from [9]. Zaslavsky used both techniques to prove the results that imply (2.1.3).

We switch gears a bit at the end of this chapter. A common question to ask about a characteristic polynomial is: When are the roots integral? Supersolvability is a matroid property that implies that the roots are integral. We describe which biased expansions of biased graphs have supersolvable biased matroids.

2.2 A Mini-Course on the Characteristic and Chromatic Polynomials

In this section we give various results about the characteristic and chromatic polynomials. See [6] for information about the characteristic polynomial of a geometric lattice, [9] for the development of the characteristic and chromatic polynomials that are related to biased graphs, and [3] for an introduction to hyperplane arrangements.

The *characteristic polynomial* of a geometric lattice L is defined to be

$$p_L(\lambda) = \sum_{x \in L} \mu_L(\hat{0}, x) \lambda^{r(L)-r(x)},$$

where μ is the Möbius function of L [6, Section 7.1], and r is the rank function of L.

Let $\mathcal{A}$ be a hyperplane arrangement in an n-dimensional vector space, and let L be its intersection lattice (ordered by reverse inclusion). The *characteristic polynomial* of $\mathcal{A}$ is defined as

$$p_{\mathcal{A}}(\lambda) = \sum_{x \in L} \mu_L(\hat{0}, x) \lambda^{\dim(x)}$$

if $\mathcal{A}$ does not include the degenerate hyperplane. Otherwise, $p_{\mathcal{A}}(\lambda) = 0$. In general, $p_{\mathcal{A}}(\lambda)$ and $p_L(\lambda)$ need not agree. However, if the degenerate hyperplane is not in $\mathcal{A}$, then

$$p_{\mathcal{A}}(\lambda) = \lambda^{n-r(L)} p_L(\lambda). \tag{2.2.1}$$

Let M be a matroid whose lattice of flats is L. To define the *characteristic* *polynomial* of M, we follow [6] and say that

$$p_M(\lambda) = \begin{cases} p_L(\lambda) & \text{if } M \text{ has no loops, and} \\ 0 & \text{otherwise.} \end{cases}$$

Let Γ be a graph, and let $\chi_\Gamma(\lambda)$ be its chromatic polynomial. Then

$$\chi_\Gamma(\lambda) = \lambda^{c(\Gamma)} p_{G(\Gamma)}(\lambda)$$

[6, Proposition 7.5.1], where $G(\Gamma)$ is the matroid of Γ. After applying the Boolean expansion formula for $p_{G(\Gamma)}$ [6, Proposition 7.2.1], we find that

$$\chi_\Gamma(\lambda) = \sum_{S \subseteq E(\Gamma)} \lambda^{c(S)} (-1)^{|S|}.$$

Similar formulas holds for the bias matroid of a biased graph Ω. In [9, Section 3], the chromatic polynomial of Ω is defined as

$$\chi_\Omega(\lambda) = \sum_{S \subseteq E(\Omega)} \lambda^{b(S)} (-1)^{|S|}.$$

Then [9, Theorem 5.1] says that

$$\chi_\Omega(\lambda) = \lambda^{b(\Omega)} p_{G(\Omega)}(\lambda). \tag{2.2.2}$$

Consider a gain graph Φ whose gain group is a multiplicative subgroup of a skew field, and let Φ have n vertices. Construct $\mathcal{H}(\Phi)$ as described in Figure 2.1.1. According to [11, Corollary 2.2], the lattice of flats of the bias matroid $G(\Phi)$, call it L, is isomorphic to the intersection lattice of $\mathcal{H}(\Phi)$ (ordered by reverse inclusion). From

(2.2.1),

$$p_{\mathcal{H}(\Phi)}(\lambda) = \lambda^{n-r(G(\Phi))} p_{L}(\lambda);$$

and from (2.2.2),

$$\chi_{\Phi}(\lambda) = \lambda^{b(\Phi)} p_{G(\Phi)}(\lambda).$$

$\big($For notational ease, we write $\chi_{\Phi}(\lambda)$ instead of $\chi_{\langle\Phi\rangle}(\lambda)$.$\big)$ According to [8, Theorem 2.1(j)], $b(\Phi) = n - r\big(G(\Phi)\big)$. If Φ contains a balanced loop (which means that $\mathcal{H}(\Phi)$ contains the degenerate hyperplane), then $\chi_{\Phi}(\lambda) = p_{\mathcal{H}(\Phi)}(\lambda) = 0$. Otherwise, $G(\Phi)$ has no loops, so $p_{L}(\lambda) = p_{G(\Phi)}(\lambda)$. Thus

$$\chi_{\Phi}(\lambda) = p_{\mathcal{H}(\Phi)}(\lambda). \tag{2.2.3}$$

A balanced counterpart of $\chi_{\Omega}(\lambda)$ is the *balanced chromatic polynomial*

$$\chi^{b}_{\Omega}(\lambda) = \sum_{\substack{S \subseteq E(\Omega) \\ S \text{ balanced}}} \lambda^{b(S)} (-1)^{|S|}.$$

As with the chromatic polynomial of a graph, the chromatic polynomials of a gain graph have a coloring interpretation. We briefly describe how to color a gain graph Φ that has a finite gain group $\mathfrak{G}$. For more details, see [9, Section 4]. To color Φ in k colors, we assign to each vertex an element of the color set

$$C_k = \big(\{1, \ldots, k\} \times \mathfrak{G}\big) \cup \{0\}.$$

If 0 is never used, the coloring is *zero-free*. An edge is *improper* if it has the form of an edge in Figure 2.2.1. A coloring is *proper* if it has no improper edges. The important theorem about gain-graph coloring is the following:

Theorem 2.2.1 ([9, Theorem 4.2]). *Let Φ be a gain graph, and let k be a nonnegative* *integer. Then the number of proper colorings of Φ in k colors equals $\chi_{\Phi}(|\mathfrak{G}|k+1)$ and the number of zero-free proper colorings equals $\chi^b_{\Phi}(|\mathfrak{G}|k)$.*

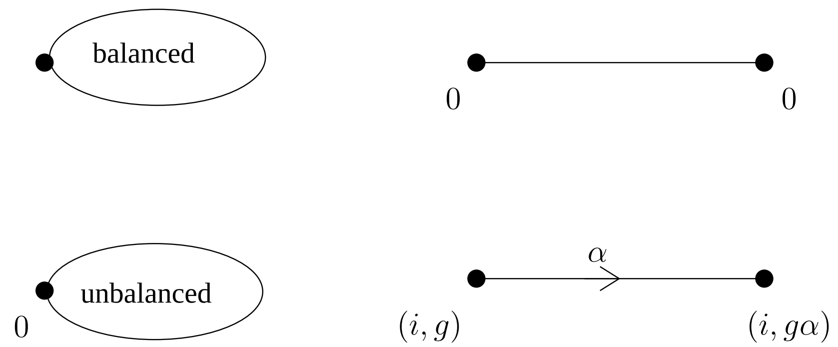


Figure 2.2.1: These are the types of improper edges in gain-graph coloring. The label at a vertex represents its color.

2.3 Biased Expansions of Biased Graphs

2.3.1 Group Expansions of Gain Graphs and the Chromatic Polynomial

Let Φ be a gain graph with gain group $\mathfrak{H}$, and let $\mathfrak{G}$ be a group. A $\mathfrak{G}$-*expansion* of Φ, denoted by $\mathfrak{G}\Phi$, is a gain graph with gain group $\mathfrak{G} \times \mathfrak{H}$ that is derived from Φ by replacing each edge of Φ with gain h by $\#\mathfrak{G}$ edges, one bearing each possible gain value (g_i, h) for $g_i \in \mathfrak{G}$.

Corollary 2.3.1. *Let Φ be a gain graph with n vertices, and let $\mathfrak{G}$ be a group with m elements. Then*

$$\chi^b_{\mathfrak{G}\Phi}(\lambda) = m^n \chi^b_{\Phi}\left(\frac{\lambda}{m}\right). \tag{2.3.1}$$

Consequently,

$$\chi_{(\mathfrak{G}\Phi)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda - 1}{m} + 1\right). \tag{2.3.2}$$

As explained in Section 2.1, Ehrenborg and Readdy [1, Theorem 3.2] proved a hyperplane analogue of (2.3.2) when Φ has gains in $\mathbb{R}^*_{>0}$, the multiplicative group of the positive real numbers, and $\mathfrak{G}$ is the group generated by a primitive mth root of unity. When the gain group of Φ is trivial, Zaslavsky [9, Examples 3.6 and 4.6] proved (2.3.1) explicitly and (2.3.2) implicitly.

Corollary 2.3.1 is actually a special case of Theorem 2.3.3, its biased-graph generalization. The proof of Theorem 2.3.3 is algebraic. To illustrate gain-graph coloring, however, we first provide a combinatorial proof of (2.3.1) when the gain group of Φ is finite. To prove his special case of (2.3.1), Zaslavsky used both the algebraic and coloring techniques.

Proof of (2.3.1) when Φ has a finite gain group. Throughout this proof we apply Theorem 2.2.1. If Φ has a balanced loop, then

$$\chi^b_{\mathfrak{G}\Phi}(\lambda) = m^n \chi^b_{\Phi}\left(\frac{\lambda}{m}\right) = 0.$$

Recall that an unbalanced loop is an improper edge only if its vertex is colored 0. Since this is not a possibility in a zero-free proper coloring, we may assume that Φ has no loops.

Assume the gain group of Φ has p elements. Set $\lambda = pk$, and count zero-free proper colorings of $\mathfrak{G}\Phi$ in k colors. Assume that vertices v_1 and v_2 are adjacent in $\mathfrak{G}\Phi$, and color them $\big(k_1, (q_1, h_1)\big)$ and $\big(k_2, (q_2, h_2)\big)$, respectively. Consult Figure 2.3.1. Since $q_1 = gq_2$ for some $g \in \mathfrak{G}$, a proper coloring of $\mathfrak{G}\Phi$ is achieved only if $k_1 \neq k_2$ or if $k_1 = k_2$ and $h_2 \neq h_1 h$. But these are the rules for properly coloring the corresponding vertices of Φ in k colors. To make a zero-free proper coloring of $\mathfrak{G}\Phi$ in k colors, therefore, simply make a zero-free proper coloring of Φ in k colors. Then, for each of the n vertices, arbitrarily choose one of the m values for the $\mathfrak{G}$-coordinate of the color. In other words,

$$\chi^b_{\mathfrak{G}\Phi}(mpk) = m^n \chi^b_{\Phi}(pk).$$

Since this is a polynomial equation valid for all $k \in \mathbb{Z}_{>0}$, it is an identity. $\square$

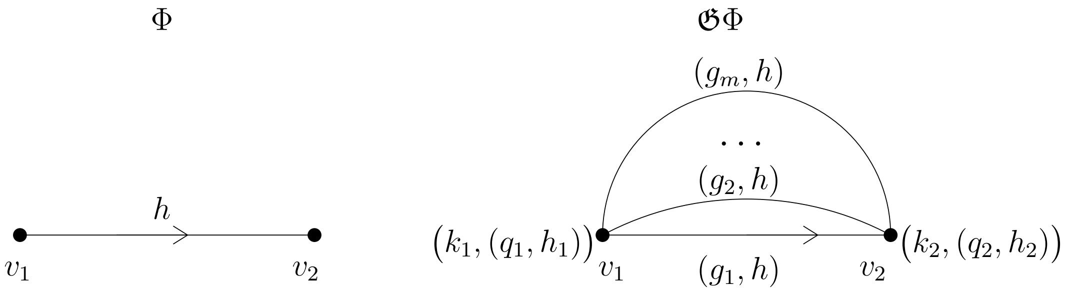


Figure 2.3.1: An edge in Φ and its corresponding edges in $\mathfrak{G}\Phi$.

Recall that the equations of the hyperplanes in a frame arrangement have one of the forms that are shown in Figure 2.1.1.

Corollary 2.3.2 ([1, Theorem 3.2])**.** *Let $\mathcal{A}$ be a frame arrangement in $\mathbb{R}^n$ where the coefficients of the hyperplane equations are positive real numbers. Then*

$$p_{(D_m(\mathcal{A}))^\bullet}(\lambda) = m^n p_{\mathcal{A}^\bullet}\left(\frac{\lambda - 1}{m} + 1\right). \tag{2.3.3}$$

Proof. Let Φ be the gain graph that encodes $\mathcal{A}$. Let ζ be a primitive mth root of unity, and let Φ_m be the gain graph that encodes $D_m(\mathcal{A})$. So an edge in Φ with gain α is associated with m edges in Φ_m with gains α, $\zeta\alpha$, $\ldots$, $\zeta^{m-1}\alpha$.

Since $\mathcal{A}$ does not contain the degenerate hyperplane, neither Φ nor Φ_m contain a balanced loop. If a vertex supports exactly one unbalanced loop, then omitting it does not affect that vertex in $\Phi^\bullet$. If a vertex supports more than one unbalanced loop, then $\mathcal{A}$ contains duplicate coordinate hyperplanes. This does not affect the characteristic polynomial. So we may assume that $\mathcal{A}$ and $D_m(\mathcal{A})$ are loopless.

By (2.2.3), we need to show that

$$\chi_{(\Phi_m)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda - 1}{m} + 1\right).$$

According to (2.3.2),

$$\chi_{(\langle\zeta\rangle\Phi)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda - 1}{m} + 1\right).$$

So we only need to show that

$$\chi_{(\Phi_m)^\bullet}(\lambda) = \chi_{(\langle\zeta\rangle\Phi)^\bullet}(\lambda).$$

This is true if Φ_m and $\langle\zeta\rangle\Phi$ are isomorphic. The graph isomorphism is easy to find: it is the identity on vertices, and it maps the edge of $\langle\zeta\rangle\Phi$ with gain (ζ^h, α) to the edge of Φ_m that has gain $\zeta^h\alpha$. Let

$$\zeta^{h_1}\alpha_1, \ldots, \zeta^{h_k}\alpha_k$$

be the gains of the edges of a circle C in Φ_m. This circle is balanced if and only if

$$\zeta^{h_1+\cdots+h_k}\alpha_1\alpha_2\cdots\alpha_k = 1.$$

But $\alpha_i > 0$, so the circle is balanced if and only if

$$\zeta^{h_1+\cdots+h_k} = \alpha_1\alpha_2\cdots\alpha_k = 1.$$

Thus C is balanced if and only if

$$(\zeta^{h_1}, \alpha_1), \ldots, (\zeta^{h_k}, \alpha_k)$$

are the gains of the edges of a balanced circle in $\langle\zeta\rangle\Phi$. $\square$

2.3.2 Biased Expansions of Biased Graphs and the Chromatic Polynomial

Biased graphs arose as a combinatorial generalization of gain graphs. So we ask, what is the biased-graph generalization of a group expansion of a gain graph?

A *biased expansion of a biased graph* Ω is a biased graph Λ together with a *projection* mapping $\pi : ||\Lambda|| \to ||\Omega||$ satisfying:

(BE1) π is the identity on vertices, and $\pi^{-1}(e) \neq \emptyset$ for each $e \in E(\Omega)$;

(BE2) (Balanced Circle Lifting Property) for each balanced circle $C = e_1e_2\cdots e_l$ in $\mathcal{B}(\Omega)$ and each $\tilde{e}_1 \in \pi^{-1}(e_1), \ldots, \tilde{e}_{l-1} \in \pi^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in \pi^{-1}(e_l)$ for which $\tilde{e}_1\tilde{e}_2\cdots\tilde{e}_l$ is in $\mathcal{B}(\Lambda)$; and

(BE3) (Balanced Circle Projection Property) if a circle $\tilde{C} \in \mathcal{B}(\Lambda)$, then $\pi(\tilde{C}) \in \mathcal{B}(\Omega)$.

A consequence of the Balanced Circle Projection Property is:

(BE4) no balanced digon of Λ projects to a single edge of Ω.

A *lift* of $S \subseteq E(\Omega)$ is any $\tilde{S} \subseteq E(\Lambda)$ such that $\pi(\tilde{S}) = S$ and $\pi|_{\tilde{S}}$ is injective. We will continue to use the notation $\tilde{S}$ for a lift of S.

The *fiber* of an edge e is the set $\pi^{-1}(e)$. An *m-fold* biased expansion of a biased graph Ω is a biased expansion where each edge fiber has m elements. Such an expansion is *regular* and has *multiplicity* m. We denote it by $m \cdot \Omega$.

I chose the definition of a biased expansion of a biased graph because it enables a combinatorial generalization of Corollary 2.3.1. Also, it generalizes Zaslavsky's definition of a biased expansion of a graph.

Notice that $\langle \mathfrak{G}\Phi \rangle$ is a $|\mathfrak{G}|$-fold biased expansion of the biased graph $\langle \Phi \rangle$.

Here is the generalization of Corollary 2.3.1.

Theorem 2.3.3. *Let Λ be an m-fold biased expansion of Ω, and assume that Ω has n vertices. Then*

$$\chi^b_\Lambda(\lambda) = m^n \chi^b_\Omega \left(\frac{\lambda}{m} \right) \tag{2.3.4}$$

Consequently,

$$\chi_{\Lambda^\bullet}(\lambda) = m^n \chi_{\Omega^\bullet} \left(\frac{\lambda - 1}{m} + 1 \right) \tag{2.3.5}$$

Proof. We begin with a proof of (2.3.4). This proof closely mimics Zaslavsky's proof for group expansions of graphs [9, Example 3.6]. Consider the following string of equalities:

$$\sum_{\substack{S \subseteq E(\Lambda) \\ S \text{ balanced}}} \lambda^{b(S)}(-1)^{|S|} = \sum_{\substack{S \subseteq E(\Lambda) \\ S \text{ balanced}}} \lambda^{b(\pi(S))}(-1)^{|\pi(S)|} = \sum_{\substack{T \subseteq E(\Omega) \\ T \text{ balanced}}} m^{n-b(T)} \lambda^{b(T)} (-1)^{|T|}.$$

By definition, the first expression is $\chi^b_\Lambda(\lambda)$ and the last expression is $m^n \chi^b_\Omega\left(\frac{\lambda}{m}\right)$. To prove the first equality, we show that $\Lambda{:}S$ and $\Omega{:}\pi(S)$ are isomorphic. That π is an isomorphism is clear, except for showing that the edge-set mapping is injective. Let e_1 and e_2 be in S. If $\pi(e_1) = \pi(e_2)$, then e_1e_2 is a digon in Λ. By (BE4), e_1e_2 is unbalanced, a contradiction. Thus $|S| = |\pi(S)|$, and S and $\pi(S)$ have the same number of components. The first equality is a consequence of these facts, together with the fact that $\pi(S)$ is balanced, which follows from the Balanced Circle Projection

Property.

For the second equality to hold, we must show that a balanced subset T of $E(\Omega)$ has $m^{n-b(T)}$ balanced lifts. Specify a maximal forest F of T. Each balanced lift of T includes a lift of F. Lift F to $\tilde{F}$. This can be done in $m^{n-b(T)}$ ways. By the Balanced Circle Lifting Property (BE2), there is at most one balanced lift of T containing $\tilde{F}$. Since $\tilde{F}$ is balanced, so is $\mathrm{bcl}(\tilde{F})$ [7, Proposition 3.1]. So $\mathrm{bcl}(\tilde{F}) \cap \pi^{-1}(T)$ is such a lift. Thus T has $m^{n-b(T)}$ balanced lifts.

Now we prove Equation (2.3.5). Since unbalanced loops do not affect the balanced chromatic polynomial,

$$\chi^b_{\Lambda^\bullet}(\lambda) = m^n \chi^b_{\Omega^\bullet}\left(\frac{\lambda}{m}\right). \tag{2.3.6}$$

If each vertex in a biased graph Ω supports an unbalanced loop, then $\chi_\Omega(\lambda) = \chi^b_\Omega(\lambda - 1)$ [9, Equation 11.1]. Thus,

$$\chi_{\Lambda^\bullet}(\lambda + 1) = m^n \chi_{\Omega^\bullet}\left(\frac{\lambda}{m} + 1\right).$$

□

Problem 2.3.4. Generalize Theorem 2.3.3 to other polynomials that appear in [9] (for example, the dichromatic polynomial).

2.3.3 What Lies Between Group Expansions of Gain Graphs and Biased Expansions of Biased Graphs?

There are two obvious classes of biased graphs that lie between regular biased expansions of biased graphs and group expansions of gain graphs: group expansions of biased graphs and regular biased expansions of gain graphs.

A biased expansion of a biased graph Ω is a *biased expansion of a gain graph* if Ω is the biased graph of a gain graph.

A *biased graph with gains*, $\Upsilon = (\Omega, \phi)$, consists of a biased graph, $\Omega = (\Gamma, \mathcal{B})$, and a gain mapping ϕ, where ϕ is defined exactly as in a gain graph. Associated with Υ is a class $\mathcal{B}(\Upsilon)$ of balanced circles. Let B be a circle of Γ. Then $B \in \mathcal{B}(\Upsilon)$ if $\phi(\mathcal{B}) = 1$ and $B \in \mathcal{B}$. $\big($So $\mathcal{B}(\Upsilon) = \mathcal{B}(\Phi) \cap \mathcal{B}(\Omega)$.$\big)$ Denote the biased graph $\big(\Gamma, \mathcal{B}(\Upsilon)\big)$ by $\langle \Upsilon \rangle$.

Let $\mathfrak{G}$ be a group. A particular example of a biased graph with gains is a $\mathfrak{G}$-*expansion of a biased graph* Ω, denoted $\mathfrak{G}\Omega$. To construct the edges of $\mathfrak{G}\Omega$, replace each edge of Ω by $\#\mathfrak{G}$ edges, one bearing each possible gain value. A circle of $\langle \mathfrak{G}\Omega \rangle$ is balanced if it has gain 1 and is a lift of a balanced circle of Ω. Technically, $\mathfrak{G}\Omega = (\Omega', \phi)$ consists of the biased graph Ω' and the gain mapping ϕ where $E(\Omega')$ consists of the edges that replace the edges of Ω, $\mathcal{B}(\Omega')$ consists of all lifts of elements of $\mathcal{B}(\Omega)$, and ϕ is defined by the gains attributed to the new edges. Technically, $\langle \mathfrak{G}\Omega \rangle$ is a biased expansion of Ω, but sometimes we call $\mathfrak{G}\Omega$ a biased expansion. If $\mathfrak{G}$ is the sign group, we write $\pm\Omega$ for $\mathfrak{G}\Omega$.

Theorem 2.3.5. *Let Λ be a biased expansion of Ω. If Λ is both a group expansion of a biased graph for some finite group and a biased expansion of a gain graph, then Λ is a group expansion of a gain graph.*

Proof. Let $\mathfrak{G}$ be a finite group. Assume that $\Lambda = \langle \mathfrak{G}\Omega \rangle$ and that Λ is a $|\mathfrak{G}|$-fold biased expansion of a gain graph Φ. In both cases, Λ is a $|\mathfrak{G}|$-fold expansion, so $||\Omega|| = \big|\big|\langle \Phi \rangle\big|\big|$.

By the Balanced Circle Lifting and Projection Properties, a circle of Ω has a balanced lift in Λ if and only if it is balanced. Thus balance in Λ determines balance in Ω. The same is true for $\langle \Phi \rangle$. This proves that $\Omega \cong \langle \Phi \rangle$. It follows that $\Lambda \cong \langle \mathfrak{G}\Phi \rangle$. $\square$

To prove results about group expansions of biased graphs, we need the concept of switching. Switching is an important technique in the theory of gain graphs [7, Section 5]. The definition of switching in [7] works in the more general context of biased graphs with gains. Let $\Upsilon = (\Omega, \phi)$ be a biased graph with gains in $\mathfrak{G}$. Define a function $\lambda : V(\Upsilon) \to \mathfrak{G}$. *Switching* by λ means replacing $\phi(e)$ by $\phi^\lambda(e) = \lambda(v)^{-1}\phi(e)\lambda(w)$, where e is oriented from v to w. The switched graph Υ^λ is called *switching equivalent* to Υ.

Lemma 2.3.6. $\langle \Upsilon^\lambda \rangle = \langle \Upsilon \rangle$.

Proof. Let $C \in \mathcal{B}\big(\langle \Upsilon^\lambda \rangle\big)$. By definition, $C \in \mathcal{B}(\Omega)$ and $\phi^\lambda(C) = 1$. By [7, Lemma 5.2], the balanced counterpart to this lemma, $\phi(C) = 1$ as well. By definition, $C \in \mathcal{B}\big(\langle \Upsilon \rangle\big)$. ☐

Lemma 2.3.7. *Let $S \subseteq E(\Upsilon)$ be balanced. Then ϕ switches to the identity gain on S.*

Proof. We may think of $\Upsilon{:}S$ as a gain graph. By [7, Lemma 5.3], ϕ switches to the identity gain on $\Upsilon{:}S$. ☐

Problem 2.3.8. Generalize gain-graph coloring to coloring of biased graphs with gains.

2.4 Supersolvability

A question that is commonly asked about the characteristic polynomial of a matroid is: When are the roots integral? In [4], Stanley introduced a matroid property called supersolvability that provides a partial answer. A matroid is *supersolvable* if it has a complete chain of modular flats. Stanley showed that supersolvable matroids have roots that are positive integers [4, Theorem 4.1]. For a graph Γ, he also showed that the graphic matroid $G(\Gamma)$ is supersolvable if and only if Γ is chordal [4, Proposition

2.8]. In [10], Zaslavsky proved the following theorem, that generalizes Stanley's result by classifying the biased graphs that have supersolvable bias matroids. The proof in [10] is incomplete, but I corrected it in [2].

Theorem 2.4.1 ([10, Theorem 2.2])**.** *Let Ω be a simply biased graph. $G(\Omega)$ is supersolvable if and only if each connected component of Ω either:*

1. *has a bias-simplicial vertex ordering; or*
2. *is a simplicial extension of one of*
   (a) $(mK_2, \emptyset)$ *where* $m \geq 2$*, or*
   (b) $\langle \pm K_3 \rangle$*, or*
   (c) $\langle +\Gamma \cup -S_k \rangle$*, where* Γ *is a chordal simple graph,* S_k *is a* k*-edge star whose vertex set lies in* $V(\Gamma)$*, and the noncentral vertices of* S_k *are a clique in* Γ*.*

Before we proceed, we define a few concepts from [10] that appear either in Theorem 2.4.1 or later in this section. In a biased graph Ω, a vertex v is *bias simplicial* if:

(s1) for each pair of edges, e and f, from v to distinct neighbors x and y, there is an xy edge which completes a balanced triangle;

(s2) for each unbalanced digon that has one endpoint at v, the other endpoint is in $U(\Omega)$; and

(s3) if v is in $U(\Omega)$, then every neighbor is in $U(\Omega)$.

We call v *simplicial* if it satisfies (s1), is not in $U(\Omega)$, and is not in an unbalanced digon. A *bias-simplicial vertex ordering* (b.s.v.o.) of Ω is a linear ordering of the vertices, say $(v_1, \ldots, v_n)$, such that each v_i is bias simplicial in $\Omega{:}\{v_1, \ldots, v_i\}$. We call Ω_0 a *simplicial extension* of Ω if Ω is an induced subgraph of Ω_0 and $V(\Omega)^c$ can be linearly ordered, say $(w_1, \ldots, w_n)$, so that each w_i is simplicial in $\Omega_0{:}\,(V(\Omega) \cup \{w_1, \ldots, w_i\})$. We call Ω a *base* of Ω_0.

Theorem 2.4.4 characterizes the biased expansions that have supersolvable bias matroids. Certainly Theorem 2.4.1 is instrumental in the proof. Notice, however, that it applies to simply biased graphs. This is slightly annoying because a biased expansion Λ of a simply biased graph Ω need not be simply biased. The problem occurs when v supports an unbalanced loop in Ω. Then it may support multiple unbalanced loops in Λ. In the proof of Lemma 2.4.2, we fix the problem by letting Λ' be Λ less all but one unbalanced loop at each vertex. (If a vertex does not support an unbalanced loop in Λ, then Λ' has no unbalanced loop there). So if Λ' has no unbalanced loop, then $\Lambda' = \Lambda$. Clearly, $G(\Lambda)$ is supersolvable if and only if $G(\Lambda')$ is supersolvable. Also, Λ has a b.s.v.o. if and only if Λ' has a b.s.v.o. (the same ordering works).

In the proofs of our supersolvability results, we assume that the biased graphs are connected. We can do this because $G(\Omega)$ is supersolvable if and only if $G(\Omega')$ is supersolvable for each component Ω' of Ω. The proof of this is straight-forward given the fact that if H is a modular copoint of $G(\Omega)$, then $E(\Omega') \cap H$ is either $E(\Omega')$ or a modular copoint of $G(\Omega')$.

Lemma 2.4.2. *Let Λ be a biased expansion of a simply biased graph Ω. Assume that $|\pi^{-1}(e)| \geq 2$ for all $e \in E(\Omega)$. $G(\Lambda)$ is supersolvable if and only if each component of Λ either:*

1. *has a bias-simplicial vertex ordering, or*
2. *is $\langle \pm K_3 \rangle$ or $(mK_2, \emptyset)$ for some $m \geq 2$.*

Proof. Sufficiency follows immediately from Theorem 2.4.1. We may assume that Λ is connected. Assume that $G(\Lambda)$ is supersolvable but does not have a b.s.v.o. Then Λ'

is a simplicial extension of one of the base graphs in Theorem 2.4.1 (2). A simplicial vertex not in the base cannot be contained in an unbalanced digon. Since each link in Λ' is in an unbalanced digon, Λ' must actually be one of the base graphs. If $\langle +\Gamma \cup -S_k \rangle$ has each link in an unbalanced digon, then $\langle +\Gamma \cup -S_k \rangle = (2K_2, \emptyset)$. Thus Λ', and hence Λ, has the specified form. □

Lemma 2.4.3. *Let Λ be a biased expansion of a simply biased graph Ω. Assume that $|\pi^{-1}(e)| \geq 2$ for all $e \in E(\Omega)$. If $G(\Lambda)$ is supersolvable, then $G(\Omega)$ is supersolvable.*

Proof. We may assume that Λ is connected. Assume that $(v_1, \ldots, v_n)$ is a b.s.v.o. of Λ. We show that it is also a b.s.v.o. of Ω. Suppose that in $\Omega{:}\{v_1, \ldots, v_i\}$, v_i is incident with links e and f and that they do not form a digon. Choose lifts $\tilde{e}$ and $\tilde{f}$. Then there exists $\tilde{g}$ such that $\tilde{e}\tilde{f}\tilde{g}$ is balanced in $\Lambda{:}\{v_1, \ldots, v_i\}$. By (BG3), efg is a balanced triangle in $\Omega{:}\{v_1, \ldots, v_i\}$.

Suppose that v_i is contained in an unbalanced digon ef in $\Omega{:}\{v_1, \ldots, v_i\}$. Call the other vertex in the digon v_j. Then, by (BG3), any lift $\tilde{e}\tilde{f}$ is unbalanced. Thus $v_j \in U(\Lambda)$. But $U(\Lambda) = U(\Omega)$ because Ω has no balanced loops.

Lastly, suppose that $v_i \in U(\Omega)$ and let v_j be one of its neighbors in $\Omega{:}\{v_1, \ldots, v_i\}$. Then $v_i \in U(\Lambda)$, so $v_j \in U(\Lambda)$. Accordingly, $v_j \in U(\Omega)$.

If Λ does not have a b.s.v.o., then Lemma 2.4.2 gives the form of Λ. If $\Lambda = \langle \pm K_3 \rangle$, then $\Omega = \langle K_3 \rangle$; and if $\Lambda = (mK_2, \emptyset)$, then $\Omega = (m_1 K_2, \emptyset)$ where $2m_1 \leq m_2$. In both cases, $G(\Omega)$ is supersolvable. □

The converse of Lemma 2.4.3 is not true. Let Ω consist of a balanced triangle with an unbalanced loop at one vertex. Let $\Lambda = \langle \mathbb{Z}_2 \Omega \rangle$. Then $G(\Omega)$ is supersolvable, but $G(\Lambda)$ is not. Theorem 2.4.4 says that the converse will fail when Ω has adjacent vertices, neither of which supports an unbalanced loop.

Theorem 2.4.4. *Let Λ be a biased expansion of a simply biased graph Ω. Assume $|\pi^{-1}(e)| \geq 2$ for all $e \in E(\Omega)$. $G(\Lambda)$ is supersolvable if and only if for each component Ω' of Ω either:*

1. *Ω' has a bias-simplicial vertex ordering and there is no pair of adjacent vertices in $U(\Omega')^c$; or*

2. *Ω' is $\langle K_3 \rangle$ (and the corresponding component of Λ is $\langle \pm K_3 \rangle$); or*

3. *Ω' is $(m_1 K_2, \emptyset)$ (and the corresponding component of Λ is $(m_2 K_2, \emptyset)$ with $2m_1 \leq m_2$).*

Proof. We may assume that Ω is connected. We begin with a proof of sufficiency. According to Theorem 2.4.1, (2) and (3) imply that $G(\Lambda)$ is supersolvable. If (1) holds and $(v_1, \ldots, v_n)$ is a b.s.v.o. of Ω, we show that it is also a b.s.v.o. of Λ. In $\Lambda{:}\{v_1, \ldots, v_i\}$, assume that v_i is incident with links $\tilde{e}$ and $\tilde{f}$ and that these edges do not form a digon. Since v_i is bias simplicial in Ω, there exists $g \in E(\Omega)$ such that $\pi(\tilde{e})\pi(\tilde{f})g$ is a balanced triangle in Ω. By the Balanced Circle Lifting Property, there exists $\tilde{g}$ such that $\tilde{e}\tilde{f}\tilde{g}$ is a balanced triangle in Λ. Now assume that $\Lambda{:}\{v_1, \ldots, v_i\}$ has an unbalanced digon $\tilde{D}$ at v_i. Let v_j be the second vertex of $\tilde{D}$. Since Ω has no balanced digons, $\tilde{D}$ projects to an unbalanced digon or to a link. If $\pi(\tilde{D})$ is an unbalanced digon, then v_j must support an unbalanced loop in Ω. Hence $v_j \in U(\Lambda)$. If $\pi(\tilde{D})$ is a link, then at least one of v_j and v_i is in $U(\Omega)$ by hypothesis. Since v_j precedes v_i in the b.s.v.o. of Ω, $v_j \in U(\Omega)$. But then $v_j \in U(\Lambda)$ too.

Now we prove necessity. Assume Ω has a link e, both of whose vertices are not in $U(\Omega)$. Since $|\pi^{-1}(e)| \geq 2$, Λ cannot have a b.s.v.o. According to Lemma 2.4.2, $\Lambda = \langle \pm K_3 \rangle$ or $\Lambda = (m_2 K_2, \emptyset)$ for some $m_2 \geq 2$. Then Ω is $\langle K_3 \rangle$ or $(m_1 K_2, \emptyset)$ where $2m_1 \leq m_2$, respectively. Finally, assume that Ω does not have a b.s.v.o. Then neither does Λ (see the proof of Lemma 2.4.3), and we just analyzed this possibility. ☐

A special case and an application of Theorem 2.4.4 are [9, Theorem 6.4] and [1, Theorem 4.1].

Corollary 2.4.5 ([9, Theorem 6.4])**.** *Assume that* Γ *is connected and* $|\mathfrak{G}| \geq 2$ *for some group* $\mathfrak{G}$*. Then* $G(\mathfrak{G}\Gamma^{(H)})$ *is supersolvable if and only if* Γ *is chordal and either* Γ *has at most two vertices, or* H^c *is a stable set of simplicial vertices, or* $|\mathfrak{G}| = 2$ *and* $\Gamma = K_3$ *and* $H = \emptyset$*.*

Corollary 2.4.6 ([1, Theorem 4.1])**.** *Let* $\mathcal{A}$ *be a frame arrangement that includes all coordinate hyperplanes. If* $\mathcal{A}$ *is supersolvable, then so is the Dowlingization* $D_m(\mathcal{A})$*.*

Proof. Let Φ be the gain graph associated with $\mathcal{A}$, and let $\mathfrak{G}\Phi$ be the gain graph associated with its Dowlingization. Then Φ has an unbalanced loop at each vertex. Since the graphs in Theorem 2.4.1(2) do not have loops, $\langle\Phi\rangle$ must have a b.s.v.o. By Theorem 2.4.4, $G(\mathfrak{G}\Phi)$ is supersolvable too. □

In [1], the statement of Corollary 2.4.6 is incorrect. The result is stated for frame arrangements (which, as defined in [1], need not contain the coordinate hyperplanes). The proof uses the coordinate hyperplanes, however, so the authors intended them to be part of the result.

Homework Problem 2.4.7. For each result in this section, find the corresponding result for the lift matroid.

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Chapter 3

When are Biased Expansions Actually Group Expansions?

3.1 Introduction

In the theory of biased graphs, many results are motivated by the case in which gains (group elements) label a graph's edges. In fact, biased graphs are a combinatorial abstraction of gain graphs. Similarly, biased expansions of graphs are a generalization of group expansions; and biased expansions of biased graphs generalize group expansions of biased graphs. An obvious question is: Does the bias in most biased graphs come from gains?

Nongroup examples of biased graphs abound, but there is no structural result that determines whether or not a biased graph is gain biased. However, such a result does exist for biased expansions of graphs.

Theorem 3.1.1 ([5, Theorem 4.1]). *Every biased expansion of a 3-connected graph of order at least 4 is a group expansion.*

This result is false in general.

Theorem 3.1.2 ([5, Corollary 5.6]). *Every 2-connected but 2-separable simple graph has a nongroup biased expansion of every multiplicity $\gamma \geq 4$.*

So biased expansions of sufficiently connected balanced biased graphs are actually group expansions. Can we characterize group expansions of arbitrary biased graphs? Currently, I can only offer some partial results. I do know that biased graphs with 3-connected underlying graphs may have nongroup biased expansions (see, for instance, Example 3.3.9.) I also know a couple of interesting properties of group expansions of a biased graph Ω. For example, consider the edges that have unit gain. The set of these edges shows that $E(\Omega)$ has a lift with the property that unbalanced circles project to unbalanced circles and balanced circles project to balanced circles. I call biased expansions with this property *equibalanced*. So group expansions are equibalanced. In fact, group expansions satisfy a stronger property, which I call *strong equibalance*. The bulk of this chapter is about equibalance (Section 3.3) and strong equibalance (Section 3.4). Unfortunately, neither property characterizes group expansions.

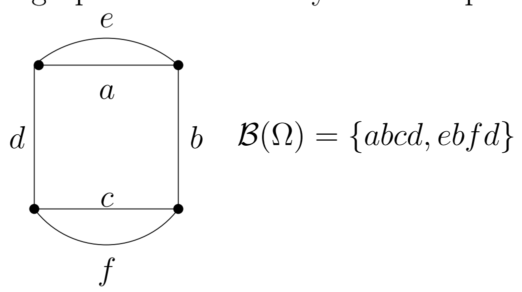
3.2 Regular Biased Expansions

Since we are interested in group expansions of biased graphs, most of this chapter is restricted to regular biased expansions. For any $m \geq 1$ and biased graph Ω, an m-fold biased expansion exists; just expand Ω by a group of order m.

Let Ω be a biased graph with an inseparable underlying graph. Unlike in the balanced case, a biased expansion of Ω need not be regular. If Ω is an unbalanced circle, for example, the cardinalities of the edge fibers may all be different.

Though biased expansions need not be regular in general, the expansions of certain biased graphs are necessarily regular. This is true, for instance, of $\langle -K_4 \rangle$. $\big(\langle -K_4 \rangle$ is notation for the gain graph (K_4, ϕ) where $\phi : E \to \{+, -\}$ is defined by $\phi(e) = -.\big)$ If a biased graph has the property that any two edges are in a common balanced circle, the Circle Lifting Property guarantees that all biased expansions are regular. This is not a necessary condition for forced regularity, however. All biased expansions of the biased graph in Figure 3.2.1 are regular but edges a and e do not lie in a common balanced circle.

Figure 3.2.1: A biased graph for which every biased expansion is regular.



Lemma 3.2.1. *Let Ω be a biased graph, and let e and f be edges. Then $\left|\pi^{-1}(e)\right| = \left|\pi^{-1}(f)\right|$ in every biased expansion of Ω if and only if there is a chain of balanced circles $C_1, \ldots, C_k$ such that $e \in C_1$, $f \in C_k$, and $C_i \cap C_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, k-1\}$.*

Proof. By the Circle Lifting Property, the fibers of all the edges in a balanced circle have the same cardinality. This proves sufficiency. To prove necessity, we define

$$S = \big\{x \in E(\Omega) \mid \text{ there is a chain of balanced circles, } C_1, \ldots, C_k, \text{ such that } e \in C_1,\ x \in C_k, \text{ and } C_i \cap C_{i+1} \neq \emptyset \text{ for each } i \in \{1, \ldots, k-1\}\big\}.$$

Suppose that e is in no balanced circle. Let Λ be the biased expansion whose edge set is the same as $E(\Omega)$, except that e is replaced by two edges, and whose set of balanced circles is $\mathcal{B}(\Omega)$. It is easy to verify that Λ is a biased expansion. Then $|\pi^{-1}(e)| = 2$ and $|\pi^{-1}(f)| = 1$, a contradiction. Thus e is in a balanced circle, and so $e \in S$.

We need to show that $f \in S$. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be finite groups with different orders. We use these groups to construct a biased expansion Λ. To construct $E(\Lambda)$, replace each edge in S by $|\mathfrak{G}_1|$ edges, one bearing each possible gain value in $\mathfrak{G}_1$; and replace each edge in S^c by $|\mathfrak{G}_2|$ edges, one bearing each possible gain value in $\mathfrak{G}_2$. Observe that if $\tilde{C}$ is a lift of an element of $\mathcal{B}(\Omega)$, then either all edges of C are in S or all edges of C are in S^c. Thus all edges in the fibers of the elements of C have gains in the same group. It makes sense, therefore, to talk about the gain of $\tilde{C}$. We finish defining Λ by declaring that

$$\mathcal{B}(\Lambda) = \{\tilde{C} \mid C \in \mathcal{B}(\Omega) \text{ and } \tilde{C} \text{ has unit gain}\}.$$

Let π be the projection map of Λ. Once we prove that Λ is a biased expansion of Ω, the assumption that $|\pi^{-1}(e)| = |\pi^{-1}(f)|$ implies that $f \in S$.

Let $\tilde{C_1}$ and $\tilde{C_2}$ be in $\mathcal{B}(\Lambda)$, and assume that $\tilde{C_1} \cup \tilde{C_2}$ is a theta graph. Let $\tilde{C_3}$ be the third circle in this theta graph. So C_1 and C_2 are in $\mathcal{B}(\Omega)$. Since Ω is a biased graph, $C_3 \in \mathcal{B}(\Omega)$. Also, either all elements in $C_1 \cup C_2$ are in S or they are all in S^c. So the gains of all edges in the fibers of $C_1 \cup C_2$ are in the same group. We conclude that $\tilde{C_3} \in \mathcal{B}(\Lambda)$ because if two circles of a theta graph have unit gain, then so does the third circle. This proves that Λ is a biased graph.

To prove that Λ is a biased expansion, we first observe that the restriction of Λ to the fibers of a balanced circle is a group expansion. Then simple properties of group expansions imply (BE2) and (BE3). Finally, we constructed Λ so that (BE1) holds. □

Proposition 3.2.2. *Let Ω be a biased graph. All biased expansions of Ω are regular if and only if for all edges e and f, there is a chain of balanced circles $C_1, \ldots, C_k$ such that $e \in C_1$, $f \in C_k$, and $C_i \cap C_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, k-1\}$.*

3.3 Equibalanced Biased Expansions

Let Ω be a biased graph, let $S \subseteq E(\Omega)$, and let Λ be a biased expansion of Ω. A lift $\tilde{S}$ of S is *equibalanced* if all unbalanced circles of $\tilde{S}$ project to unbalanced circles of S. If $E(\Omega)$ has an equibalanced lift, we say that Λ is *equibalanced*. Incidentally, the definition of biased expansion guarantees that balanced circles of $\tilde{S}$ project to balanced circles of S.

It is clear that group expansions of biased graphs are equibalanced: let $\widetilde{E(\Omega)}$ be the set of edges with unit gain, for example. Does the concept of equibalance help determine which biased expansions of biased graphs are actually group expansions? The answer is "yes" for 2-fold expansions (see Theorem 3.1) and "no" otherwise. For $\gamma \geq 4$, it is not difficult to produce nongroup equibalanced expansions. For example, according to Theorem 3.1.2, $\langle C_4 \rangle$ has a nongroup biased expansion of multiplicity γ. But every such expansion is equibalanced. For multiplicity 3, it is more complicated to produce nongroup biased expansions. We produce one in Example 3.4.2.

In [5, Theorem 7.1], Zaslavsky showed that, in the balanced case, all 2-fold biased expansions are group expansions. I expected this to be true in general, but it is not. According to the next theorem, a 2-fold biased expansion is a group expansion only if it is equibalanced. However, there exist examples that are not equibalanced (see Example 3.3.9).

Theorem 3.3.1. *Let Ω be a biased graph, and let $\Lambda = 2 \cdot \Omega$. Then Λ is equibalanced if and only if $\Lambda = \langle \pm\Omega \rangle$.*

Proof. The proof of sufficiency is clear. To prove necessity, choose an equibalanced lift of $E(\Omega)$, say $\tilde{E}$. Give each edge in the equibalanced lift the label $+$, and label the other edges $-$. This defines $\langle \pm\Omega \rangle$. Let $\tilde{C}$ be a circle in Λ. We must show that $\tilde{C}$ is balanced if and only if the number of $-$ edges in $\tilde{C}$ is even and C is balanced in Ω.

To do so, we induct on $f(\tilde{C})$, the number of $-$ edges in $\tilde{C}$.

Assume that $f(\tilde{C}) = 0$. Equivalently, $\tilde{C} \subseteq \tilde{E}$. Then, since $\tilde{E}$ is equibalanced, $f(\tilde{C}) = 0$ and C is balanced $\iff$ $\tilde{C} \subseteq \tilde{E}$ and C is balanced $\iff$ $\tilde{C}$ is balanced.

Assume that $f(\tilde{C}) > 0$. Let $\tilde{C} = \tilde{e}_1 \cdots \tilde{e}_{l-1}\tilde{e}_l$ where $\tilde{e}_l$ is labeled $-$. Let $\tilde{e}_l^*$ be the other edge projecting to e_l. So $f(\tilde{e}_1 \cdots \tilde{e}_{l-1}\tilde{e}_l^*) = f(\tilde{C}) - 1$. Then

$$\begin{aligned}
\tilde{C} \text{ is balanced} &\iff \tilde{e}_1 \cdots \tilde{e}_{l-1}\tilde{e}_l^* \text{ is unbalanced and } C \text{ is balanced} \\
&\iff f(\tilde{e}_1 \cdots \tilde{e}_{l-1}\tilde{e}_l^*) \text{ is odd and } C \text{ is balanced} \qquad \text{(by induction)} \\
&\iff f(\tilde{C}) \text{ is even and } C \text{ is balanced.}
\end{aligned}$$

□

The proof of Theorem 3.3.1 is Zaslavsky's proof of the balanced case [5, Theorem 7.1] adapted to the unbalanced case. His result follows from our Theorem because of the following lemma, which implies that biased expansions of balanced biased graphs are equibalanced. (In the lemma, let $A = \emptyset$ and let $B = E(\Gamma)$.)

Lemma 3.3.2 ([5, Lemma 2.3])**.** *Let Ω be a biased expansion of a graph Γ. Given any subsets $A \subseteq B \subseteq E(\Gamma)$ and a balanced lift $\tilde{A}$, there is a balanced lift $\tilde{B}$ that contains $\tilde{A}$.*

Theorem 3.3.1 provides a reason to wonder which biased graphs have only biased expansions that are equibalanced. Some basic examples are biased graphs that are almost all balanced or almost all contrabalanced.

Lemma 3.3.3. *Let Ω be a biased graph whose underlying graph is inseparable. Suppose no theta graph of Ω contains both a balanced and an unbalanced circle. Then Ω is either balanced or contrabalanced.*

Proof. We use Tutte's path theorem (see [1, Theorem 4.34]), which for inseparable graphs says that if $\mathcal{L}$ is a linear class of circles and C_0, C are circles such that $C \notin \mathcal{L}$, then there is a "path of circles,"

$$C_0, C_1, \ldots, C_k = C,$$

such that $C_i \cup C_{i+1}$, $0 \leq i \leq k-1$, are theta graphs and $C_1, \ldots, C_{k-1} \notin \mathcal{L}$.

Suppose that Ω contains a balanced circle C_0 and an unbalanced circle C. By Tutte's path theorem, there is a path of circles,

$$C_0, C_1, \ldots, C_k = C,$$

so that $C_i \notin \mathcal{B}(\Omega)$ for $i > 0$ and consecutive circles form a theta graph. Since C is unbalanced, $C \cup C_{k-1}$ is a contrabalanced theta graph. Thus C_{k-1} is unbalanced. By induction, $C_1 \cup C_0$ is contrabalanced. This contradicts the assumption that C_0 is balanced. □

Proposition 3.3.4. *Let Ω be a biased graph. Suppose no theta graph of Ω contains both a balanced and an unbalanced circle. Then all biased expansions are equibalanced.*

Proof. Assume that Ω is inseparable. According to Lemma 3.3.3, Ω is either balanced or contrabalanced. The conclusion is obvious in the contrabalanced case. In the balanced case, it is implied by [5, Lemma 2.4].

If Ω is not inseparable, then each block is either balanced or contrabalanced. Combine an equibalanced lift of each block to get an equibalanced lift of Ω. □

Proposition 3.3.4 applies to biased graphs with exactly one unbalanced circle.

Proposition 3.3.5. *Let Ω be a biased graph with exactly k balanced circles. If these circles can be ordered C_1, …, C_k so that $C_i \not\subseteq C_1 \cup \cdots \cup C_{i-1}$ for each $i > 1$, then all biased expansions of Ω are equibalanced.*

Proof. Let Λ be a biased expansion of Ω. We show how to find an equibalanced lift of $C_1 \cup \cdots \cup C_k$. Then an equibalanced lift of $E(\Omega)$ is found by arbitrarily lifting all edges that are not in a balanced circle.

The proof is by induction on k. If C is the only balanced circle in Ω, choose any balanced lift of C.

Assume $k \geq 2$. By induction, there is an equibalanced lift of $C_1 \cup \cdots \cup C_{k-1}$. Let $e \in C_k \backslash (C_1 \cup \cdots \cup C_{k-1})$. Arbitrarily lift the elements of $C_k \backslash (C_1 \cup \cdots \cup C_{k-1} \cup \{e\})$. According to the Circle Lifting Property, there is a unique $\tilde{e}$ that completes a balanced lift of C_k. Now we have an equibalanced lift of $C_1 \cup \cdots \cup C_k$. □

Proposition 3.3.6. *Let Ω be a biased graph, and let $C_1, \ldots, C_k$ be all of its balanced circles. If $C_1 \cup \cdots \cup C_k$ is balanced, then all biased expansions of Ω are equibalanced.*

Proof. In Lemma 3.3.2, let $A = \emptyset$ and let $B = C_1 \cup \cdots \cup C_k$. According to the lemma, a biased expansion of Ω contains a balanced lift of B. To specify an equibalanced lift of $E(\Omega)$, arbitrarily lift the edges of $E(\Omega) \backslash B$. □

Corollary 3.3.7. *All biased expansions of a biased graph Ω are equibalanced if:*

1. *Ω has exactly one balanced circle;*
2. *Ω has exactly two balanced circles; or*
3. *Ω has exactly three balanced circles, C_1, C_2, and C_3, and these circles either form a theta graph or satisfy $C_i \not\subseteq C_j \cup C_k$ for some choice of $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. Part 1 follows from Proposition 3.3.5 immediately. In part 2, label the balanced circles C_1 and C_2. Then apply Proposition 3.3.5.

In part 3, assume that $C_i \not\subseteq C_j \cup C_k$. Order the balanced circles C_j, C_k, C_i. Then apply Proposition 3.3.5.

If, instead, the three balanced circles in part 3 form a theta graph, then apply Proposition 3.3.6. □

If a biased expansion is not equibalanced, then it is not a group expansion. Using this fact, we show that there exist nongroup biased expansions of biased graphs with 3-connected underlying graphs. This is not true in the balanced case (see Theorem 3.1.1).

Example 3.3.8. In [3, Section 7], Zaslavsky shows that K_4 is the underlying graph of precisely seven biased graphs. These are described in the table below.

	Which circles are balanced?
Ω_1	all circles
Ω_2	three circles of a theta graph
Ω_3	one triangle
Ω_4	three quadrilaterals
Ω_5	two quadrilaterals
Ω_6	one quadrilateral
Ω_7	no circles

According to Proposition 3.3.4 and Corollary 3.3.7, any biased expansion of Ω_i, $i \neq 4$, is equibalanced. Example 3.3.9 shows that this is false for Ω_4. Incidentally, $\Omega_4 = \langle -K_4 \rangle$.

Example 3.3.9. We show that for $\gamma \geq 2$ there exists $\Lambda_\gamma = \gamma \cdot \langle -K_4 \rangle$, a biased expansion that is not equibalanced. It follows that Λ_γ is not a group expansion.

First we explain how to construct Λ_γ. Consult Figure 3.3.1 for a description of Λ_3. Let the edges of K_4 be A, B, C, D, E, and F, oriented as in Figure 3.3.1. In Λ_γ, let the edges in the fiber of A be $A_0, A_1, \ldots, A_{\gamma-1}$. Do the same for the other fibers.

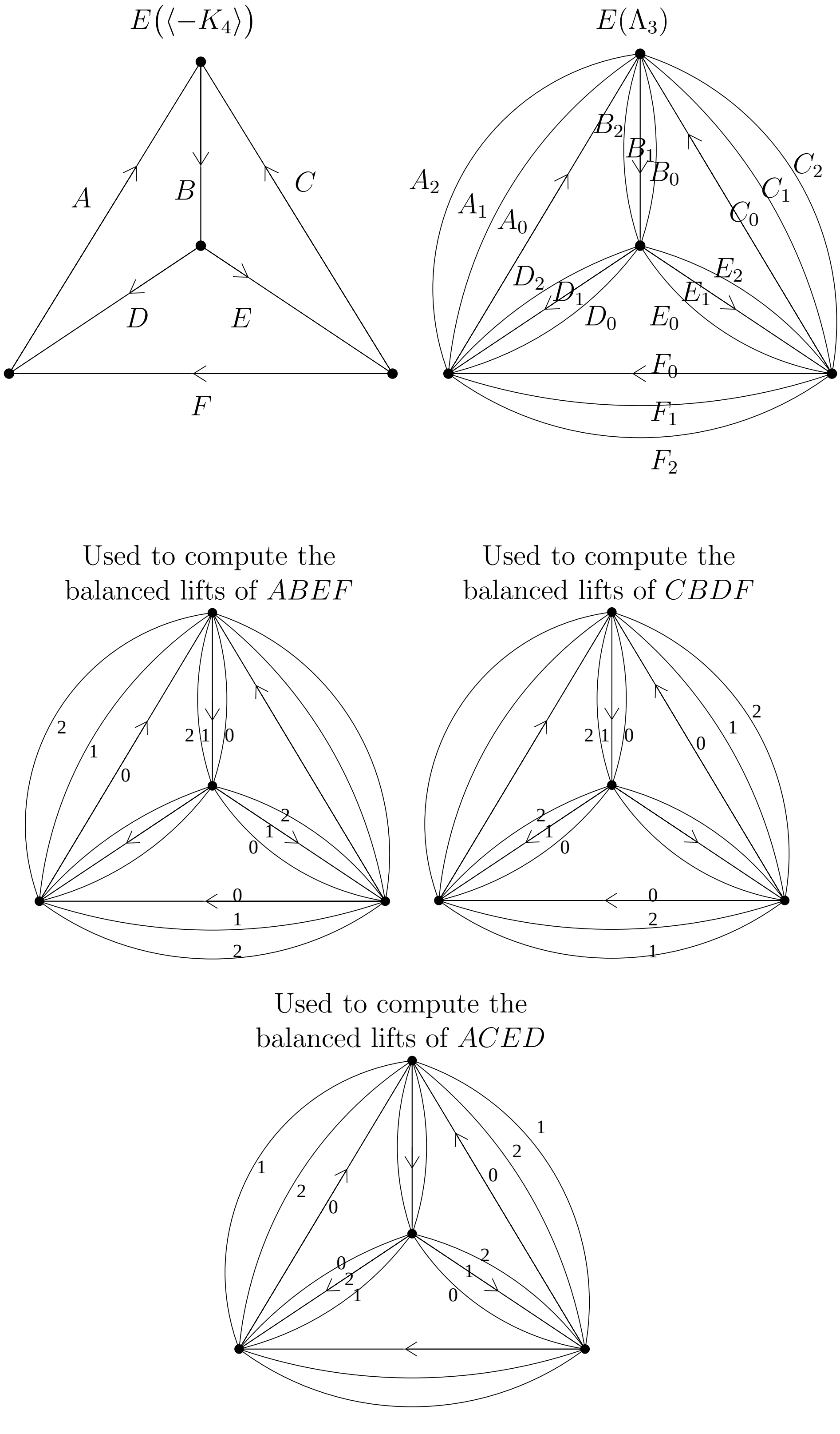


Figure 3.3.1: These figures describe the construction of Λ_3.

To define $\mathcal{B}(\Lambda_\gamma)$, we define the balanced circles that project to each quadrilateral. (There are three quadrilaterals, which we refer to as the left, right, and top quadrilaterals.) The balanced lifts of each quadrilateral can be assigned independently because there is no theta-graph condition to satisfy. The restriction of Λ_γ to the fibers of a particular quadrilateral is a $\mathbb{Z}_\gamma^+$-expansion of $\langle C_4 \rangle$. ($\mathbb{Z}_\gamma^+$ denotes the group whose operation is addition modulo γ.) We denote the gain mappings of the restrictions to the left, right, and top quadrilaterals by ϕ_L, ϕ_R, and ϕ_T respectively. Define

$$\begin{array}{lllll} \phi_L(A_i) = i, & \phi_L(B_j) = j, & \phi_L(E_k) = k, & \text{and} & \phi_L(F_l) = l; \\ \phi_R(C_m) = m, & \phi_R(B_j) = j, & \phi_R(D_p) = p, & \text{and} & \phi_R(F_l) = -l; \quad \text{and} \\ \phi_T(A_i) = -i, & \phi_T(C_m) = -m, & \phi_T(E_k) = k, & \text{and} & \phi_T(D_p) = p + 1. \end{array}$$

Accordingly,

$$\begin{aligned} \mathcal{B}(\Lambda_\gamma) = &\{A_i B_j E_k F_l \mid i + j + k + l \equiv 0 \mod \gamma\} \cup \\ &\{C_m B_j D_p F_l \mid m + j + p + l \equiv 0 \mod \gamma\} \cup \\ &\{A_i C_m E_k D_p \mid -i + m - k + (p + 1) \equiv 0 \mod \gamma\}. \end{aligned}$$

Now we show that Λ_γ is not equibalanced. Choose a balanced lift of $ABEF$, namely $A_i B_j E_k F_l$. Also, choose a balanced lift of $CBDF$ that includes B_j and F_l, say $C_m B_j D_p F_l$. Then

$$i + j + k + l \equiv 0 \mod \gamma \qquad \text{and} \qquad m + j + p + l \equiv 0 \mod \gamma.$$

Thus $i + k \equiv m + p \mod \gamma$. Circle $A_i C_m E_k D_p$ is not balanced because

$$-i + m - k + (p + 1) \equiv 1 \not\equiv 0 \mod \gamma.$$

3.4 Strongly Equibalanced Biased Expansions

In Section 3.3 we found that equibalance does not characterize group expansions of biased graphs. Here we define a stronger property. Let Λ be a biased expansion of a biased graph Ω. We say that Λ is *strongly equibalanced* if every balanced lift of a balanced edge set of Ω extends to an equibalanced lift of $E(\Omega)$.

Theorem 3.4.1. *Let $\mathfrak{G}\Omega$ be a group expansion of the biased graph Ω. Then $\langle\mathfrak{G}\Omega\rangle$ is strongly equibalanced.*

Proof. Let $S \subseteq E(\Omega)$ be balanced, and let $\tilde{S}$ be any balanced lift. By Lemma 2.3.7, there exists a switching function λ such that the edges of $\tilde{S}$ have identity gain in $(\mathfrak{G}\Omega)^\lambda$. Let $\tilde{E}$ consist of the edges of $(\mathfrak{G}\Omega)^\lambda$ that have gain 1. According to Lemma 2.3.6, $\big\langle(\mathfrak{G}\Omega)^\lambda\big\rangle = \langle\mathfrak{G}\Omega\rangle$. Thus $\tilde{E}$ is an equibalanced lift of $E(\Omega)$ that contains $\tilde{S}$. $\square$

The converse of Theorem 3.4.1 is not true; there exist nongroup strongly equibalanced biased expansions of biased graphs (see Example 3.4.3).

Example 3.3.9 revisited. We provide another proof that Λ_γ is not a group expansion. Consider the balanced circle $A_0 B_0 D_0 F_0$ in Λ_γ. We show that this circle is not contained in an equibalanced lift of $E\big(\langle -K_4\rangle\big)$. Thus Λ_γ is not strongly equibalanced and is therefore not a group expansion.

Suppose $\{A_0, B_0, E_0, F_0, D_p, C_m\}$ is an equibalanced lift. Since $C_m B_0 D_p F_0$ is balanced,

$$m + p \equiv 0 \mod \gamma.$$

And since $A_0C_mE_0D_p$ is balanced,

$$m + p + 1 \equiv 0 \mod \gamma.$$

This is a contradiction.

It is possible for a biased expansion of a biased graph to be equibalanced but not strongly equibalanced.

Example 3.4.2. Let $\gamma \geq 3$ be odd. We construct a γ-fold biased expansion of $\langle -K_4 \rangle$ as we did in Example 3.3.9, except that we redefine the gain mappings ϕ_R and ϕ_T. Here,

$$\begin{array}{lllll}
\phi_L(A_i) = i, & \phi_L(B_j) = j, & \phi_L(E_k) = k, & \text{and} & \phi_L(F_l) = l; \\
\phi_R(C_m) = m, & \phi_R(B_j) = j, & \phi_R(D_p) = p, & \text{and} & \phi_R(F_l) = l; \quad \text{and} \\
\phi_T(A_i) = -i, & \phi_T(C_m) = m, & \phi_T(E_k) = -k, & \text{and} & \phi_T(D_p) = p + 1.
\end{array}$$

Accordingly,

$$\begin{aligned}
\mathcal{B}(\Lambda_\gamma) = &\{A_iB_jE_kF_l \mid i + j + k + l \equiv 0 \mod \gamma\} \cup \\
&\{C_mB_jD_pF_l \mid m + j + p - l \equiv 0 \mod \gamma\} \cup \\
&\{A_iC_mE_kD_p \mid -i - m + k + (p + 1) \equiv 0 \mod \gamma\}.
\end{aligned}$$

Thus $A_0B_0C_{-(\gamma-1)/2}E_0D_{(\gamma-1)/2}F_0$ is an equibalanced lift of $E\big(\langle -K_4 \rangle\big)$.

The edges A_0, D_0, and F_0 form a lift of a balanced edge set. We show that they are not contained in an equibalanced lift of $E\big(\langle -K_4 \rangle\big)$, which proves that Λ_γ is not strongly equibalanced. Suppose $\{A_0, B_j, C_m, D_0, E_k, F_0\}$ is an equibalanced lift. Then

$$j + k \equiv 0 \mod \gamma, \quad m + j \equiv 0 \mod \gamma, \quad \text{and} \quad -m + k + 1 \equiv 0 \mod \gamma.$$

The first two formulas imply that $-m + k \equiv 0 \mod \gamma$, which contradicts the third formula.

We conclude this section with a family of biased expansions that are strongly equibalanced but are not group expansions. This proves that strong equibalance does not characterize group expansions.

Example 3.4.3. For $\gamma \geq 3$, let Λ_γ be the biased expansion described in Example 3.3.9, except that we redefine ϕ_T. Define

$$\begin{array}{lllll} \phi_L(A_i) = i, & \phi_L(B_j) = j, & \phi_L(E_k) = k, & \text{and} & \phi_L(F_l) = l; \\ \phi_R(C_m) = m, & \phi_R(B_j) = j, & \phi_R(D_p) = p, & \text{and} & \phi_R(F_l) = -l; \quad \text{and} \\ \phi_T(A_i) = i + 1, & \phi_T(C_m) = m - 1, & \phi_T(E_k) = -k, & \text{and} & \phi_T(D_p) = -(p + 2). \end{array}$$

Accordingly,

$$\begin{aligned} \mathcal{B}(\Lambda_\gamma) = &\{A_iB_jE_kF_l \mid i + j + k + l \equiv 0 \mod \gamma\} \cup \\ &\{C_mB_jD_pF_l \mid m + j + p + l \equiv 0 \mod \gamma\} \cup \\ &\{A_iC_mE_kD_p \mid i - m + k - p \equiv 0 \mod \gamma\}. \end{aligned}$$

We choose $\gamma \geq 3$ because we want $2 \not\equiv 0$.

This example has the following property: In a lift $\tilde{E}$ of $E(\langle -K_4 \rangle)$, if two of the quadrilaterals are balanced, so is the third. Thus $\tilde{E}$ is equibalanced if and only if two quadrilaterals in $\tilde{E}$ are balanced.

To show that Λ_γ is strongly equibalanced, we must show that every maximal balanced edge set in Λ_γ (a balanced edge set to which adding any additional edge creates imbalance) is contained in an equibalanced lift of $E(\langle -K_4 \rangle)$.

There are two types of maximal balanced edge sets: a balanced quadrilateral and a 3-edge star. Assume we have a balanced lift of a quadrilateral. Any other quadrilateral contains an edge not in the first, so the balanced lift of the first quadrilateral can be extended to a balanced lift of both quadrilaterals (as done in the proof of Proposition 3.3.5). This gives the desired equibalanced lift.

Consider the 3-edge star $\{A_i, D_p, F_l\}$. We need to find values of j, k, and m so that $\{A_i, B_j, C_m, D_p, E_k, F_l\}$ is a balanced lift of $E(\langle -K_4 \rangle)$. It is easy to see that choosing

$$m = 0, \quad j \equiv -(l+p) \mod \gamma, \qquad \text{and} \qquad k \equiv p - i \mod \gamma$$

works. The other cases are equally easy.

Showing that Λ_γ is a nongroup biased expansion requires a surprisingly small amount of information. The definitions of ϕ_L and ϕ_R are enough. The proof follows from Proposition 3.5.1.

3.5 Which Biased Graphs have Nongroup Expansions?

For graphs, the property of 3-connectedness is enough to force a biased expansion to have the structure of a group expansion. Is there a property of biased graphs that has the same effect? Certainly, the property is not 3-connectedness of the underlying graph. We conclude this chapter with a couple of results related to this issue. It is clear that I am far away from a resolution. In fact, I have not found an unbalanced biased graph for which all biased expansions are group expansions.

We extract from Example 3.4.3 the property that prevents Λ_γ from being a group expansion.

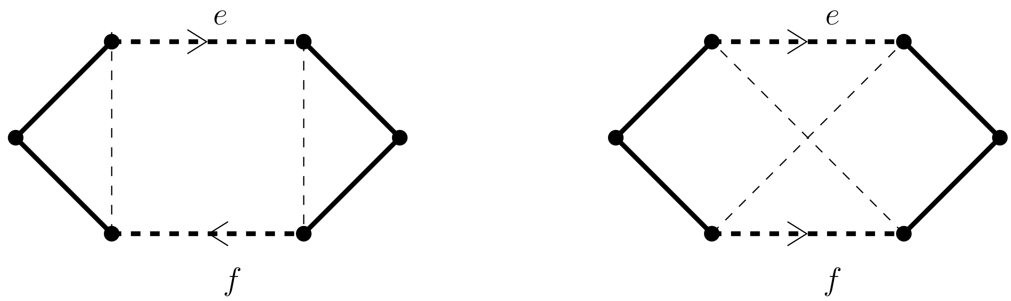


Figure 3.5.1: These are examples of $C_1 \cup C_2$ in Proposition 3.5.1. C_1 is dotted and C_2 is heavy.

Proposition 3.5.1. *Assume that Ω has two distinct balanced circles C_1 and C_2, and assume that these circles have edges e and f in common. For $\gamma \geq 3$, assume Λ is a γ-fold biased expansion of Ω that satisfies*

$$\Lambda{:}\pi^{-1}(C_i) \cong \left\langle \mathbb{Z}_\gamma^+ \langle C_i \rangle \right\rangle.$$

These isomorphisms define gain mappings

$$\phi_i : \left(\Lambda{:}\pi^{-1}(C_i)\right) \to \mathbb{Z}_\gamma^+.$$

For $x \in C_1 \cup C_2$, name the elements of its fiber $x_0, \ldots, x_{\gamma-1}$ using the rule that $\phi_1(x_i) = i$ if $x \in C_1$ and $\phi_2(x_i) = i$ if $x \in C_2 \backslash C_1$. Suppose that $\phi_2(f_i) = -i$ and that $\phi_2(x_i) = i$ for each $x \in (C_2 \cap C_1) \backslash \{f\}$. Then Λ is not a group expansion.

Proof. Fix an orientation of each edge of Ω. For notational ease, we assume that all edges in C_1 have the same orientation. Throughout this proof we write equality when we really mean congruence modulo γ.

For a 2-connected loopless graph Δ, if $\langle \mathfrak{G}\Delta \rangle$ has gains in $\mathfrak{H}$, then $\mathfrak{H}$ contains a subgroup isomorphic to $\mathfrak{G}$ [4, Theorem 2.1]. Now suppose Λ is a $\mathfrak{G}$-expansion of Ω. Then $\Lambda{:}\pi^{-1}(C_i) \cong \left\langle \mathbb{Z}_\gamma^+ \langle C_i \rangle \right\rangle$ has gains in $\mathfrak{G}$. Accordingly, $\mathfrak{G}$ contains a subgroup isomorphic to $\mathbb{Z}_\gamma^+$. Hence $\mathfrak{G} \cong \mathbb{Z}_\gamma^+$. That is, if Λ is a group expansion, it is a $\mathbb{Z}_\gamma^+$-expansion.

Suppose that Λ is a $\mathbb{Z}_\gamma^+$-expansion with gain mapping ϕ.

Because $C_1 \neq C_2$, e and f are not parallel. So each of e and f has a vertex not incident with the other edge. Use these vertices to find a switching function λ satisfying $\phi^\lambda(e_0) = \phi^\lambda(f_0) = 0$. According to Theorem 2.3.6, we may replace ϕ by ϕ^λ. That is, Theorem 2.3.6 allows us to assume that

$$\phi(e_0) = \phi(f_0) = 0.$$

Orient C_1 and C_2 in the direction that agrees with the orientation of e. There are two cases. *Case 1:* The orientation of f agrees with the orientation of C_2 (e.g., the left part of Figure 3.5.1). *Case 2:* The orientation of f opposes the orientation of C_2 (e.g., the right part of Figure 3.5.1).

For $y \in C_2$, let $\epsilon_y = 1$ if the orientations of y and C_2 agree; and $\epsilon_y = -1$ otherwise. So $\epsilon_e = 1$.

In both cases, $\{x_0 \mid x \in C_1\}$ and $\{y_0 \mid y \in C_2\}$ are balanced circles in Λ because

$$\sum_{x \in C_1} \phi_1(x_0) = \sum_{y \in C_2} \epsilon_y \phi_2(y_0) = 0.$$

Recall that $\phi(e_0) = \phi(f_0) = 0$. Hence

$$\sum_{x \in C_1 \setminus \{e,f\}} \phi(x_0) = \sum_{y \in C_2 \setminus \{e,f\}} \epsilon_y \phi(y_0) = 0. \tag{3.5.1}$$

Consider Case 1, in which $\epsilon_f = 1$. The circle $\big\{y_0 \mid y \in C_2\backslash\{e,f\}\big\} \cup \{e_1, f_1\}$ is balanced because

$$\sum_{y\in C_2\backslash\{e,f\}} \epsilon_y\phi_2(y_0) + \epsilon_e\phi_2(e_1) + \epsilon_f\phi_2(f_1) = 0 + \phi_2(e_1) + \phi_2(f_1) = 0 + 1 + (-1) = 0.$$

Also, $\big\{x_0 \mid x \in C_1\backslash\{e,f\}\big\} \cup \{e_1, f_{-1}\}$ is a balanced circle because

$$\sum_{x\in C_1\backslash\{e,f\}} \phi_1(x_0) + \phi_1(e_1) + \phi_1(f_{-1}) = 0 + 1 + (-1) = 0.$$

Therefore,

$$\sum_{y\in C_2\backslash\{e,f\}} \epsilon_y\phi(y_0) + \phi(e_1) + \phi(f_1) = \sum_{x\in C_1\backslash\{e,f\}} \phi(x_0) + \phi(e_1) + \phi(f_{-1}) = 0. \quad (3.5.2)$$

Combining (3.5.1) and (3.5.2), we find that $\phi(f_1) = \phi(f_{-1})$. But this implies that $f_1 = f_{-1}$, which is false because $\gamma \geq 3$.

In Case 2, $\epsilon_f = -1$. In this case, $\big\{y_0 \mid y \in C_2\backslash\{e,f\}\big\} \cup \{e_1, f_{-1}\}$ is a balanced circle because

$$\sum_{y\in C_2\backslash\{e,f\}} \epsilon_y\phi_2(y_0) + \epsilon_e\phi_2(e_1) + \epsilon_f\phi_2(f_{-1}) = 0 + \phi_2(e_1) - \phi_2(f_{-1}) = 0 + 1 - 1 = 0.$$

Thus

$$\sum_{y\in C_2\backslash\{e,f\}} \epsilon_y\phi(y_0) + \phi(e_1) - \phi(f_{-1}) = 0.$$

By equation (3.5.1), we find that

$$\phi(e_1) = \phi(f_{-1}). \quad (3.5.3)$$

Choose $g \in C_1\backslash\{e,f\}$. Then $\big\{x_0 \mid x \in C_1\backslash\{e,f,g\}\big\} \cup \{e_0, g_1, f_{-1}\}$ and $\big\{x_0 \mid x \in$

$C_1 \backslash \{e, f, g\}\} \cup \{e_1, g_1, f_{-2}\}$ are balanced circles. Accordingly,

$$\sum_{x \in C_1 \backslash \{e,f,g\}} \phi(x_0)+\phi(e_0)+\phi(g_1)+\phi(f_{-1}) = \sum_{x \in C_1 \backslash \{e,f,g\}} \phi(x_0)+\phi(e_1)+\phi(g_1)+\phi(f_{-2}) = 0.$$

Apply (3.5.1) and (3.5.3) to find that

$$\phi(e_0) = \phi(f_{-2}).$$

Since $\phi(e_0) = 0$, we conclude that $\phi(f_{-2}) = 0$. But this contradicts the assumptions that $\phi(f_0) = 0$ and $\gamma \geq 3$. □

Question 3.5.2. According to Theorem 3.1.1, the assumptions of Proposition 3.5.1 cannot be satisfied for balanced biased graphs whose underlying graphs are 3-connected. Why not?

Corollary 3.5.3. *If Ω has Hamiltonian bias and two Hamiltonian circles in $\mathcal{B}(\Omega)$ have at least two edges in common, then Ω has a nongroup biased expansion of every multiplicity $\gamma \geq 3$.*

least 3.5.3

In Example 3.4.3, $\langle -K_4 \rangle$ has Hamiltonian bias; and C_1 and C_2 in Proposition 3.5.1 are the left and right quadrilaterals respectively.

There is an easier way to use Hamiltonian bias to create nongroup biased expansions of multiplicity at least 4.

Proposition 3.5.4. *Suppose a biased graph Ω contains a balanced circle of length at least 4 that is in no theta graph. Let $\gamma \geq 4$. Then Ω has a nongroup biased expansion of multiplicity γ.*

Proof. Let C be the balanced circle of length at least 4. Choose a group with γ elements, say $\mathfrak{G}$. Let $\Lambda = \langle \mathfrak{G}\Omega \rangle$. Let Λ' have the structure of Λ, except that we redefine the balanced lifts of C. Let $\Lambda':\pi^{-1}(C)$ be a nongroup biased expansion of multiplicity γ. This can be done because of Theorem 3.1.2 and the assumption that C is in no theta graph. $\square$

3.6 Future Project

Let Λ be a biased expansion of a biased graph Ω. If $\Omega{:}S$ is 3-connected and balanced, we know that $\Lambda{:}\pi^{-1}(S)$ is a group expansion. For the purposes of determining which biased expansions are group expansions, it seems relevant to study the 3-connected, balanced pieces of Ω. More importantly, this project would add to the theory of biased graphs. Currently there is no biased-graphic version of Tutte's decomposition of 2-connected graphs into "3-blocks" (3-connected graphs, circuits, and multiple edges) [2, Chapter 4].

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