Biased Expansions of Biased Graphs and their Chromatic Polynomials

Lori Koban

Department of Mathematics and Computer Science

Western Carolina University

Email: lkoban@email.wcu.edu

Phone: (828)227-2484

Fax: (828)227-7240

Address: WCU, Dept. of Math & C.S., Cullowhee, NC 28723

Abstract

Label each edge of a graph with a group element. Call the labels *gains*, and call the graph with this labeling a *gain graph*. A *group expansion* of an ordinary graph is an example of a gain graph. To construct one, replace each edge of a graph by several edges, one bearing as gain each possible value in a group. We introduce the concept of a *group expansion of a gain graph*. Then we find a formula that relates the chromatic polynomials of a gain graph and its group expansions. Our main result is a generalization of this formula, in which group expansions of gain graphs are generalized to *biased expansions of biased graphs*. The inspiration for our definitions and theorems is results of Zaslavsky and of Ehrenborg and Readdy. Zaslavsky found our formula for group expansions of ordinary graphs. Ehrenborg and Readdy found a hyperplane analogue of our formula when gain graphs with real positive gains are expanded by a finite group of roots of unity.

1. Introduction

Given a geometric lattice L of rank r, there is an associated monic polynomial of degree r called the *characteristic polynomial* $p_L(\lambda)$. If L is the lattice of flats of a graphic matroid, then $p_L(\lambda)$ is closely related to the chromatic polynomial of the associated graph [?, Proposition 7.5.1]. If L is the intersection lattice of a finite collection of hyperplanes in $\mathbb{R}^n$, then $\left|p_L(-1)\right|$ is the number of regions formed by the hyperplanes [?, Section 2, Theorem A].

A *hyperplane arrangement* is a finite collection of hyperplanes in an n-dimensional vector space. We allow the *degenerate hyperplane*, which is the entire vector space. If two hyperplane arrangements are closely related, we wonder how their characteristic polynomials are related. In [?], Ehrenborg and Readdy answered this question for a *frame arrangement* $\mathcal{A}$ and its *Dowlingization* $D_m(\mathcal{A})$. For frame arrangements in $\mathbb{R}^n$ with positive real coefficients, they showed that

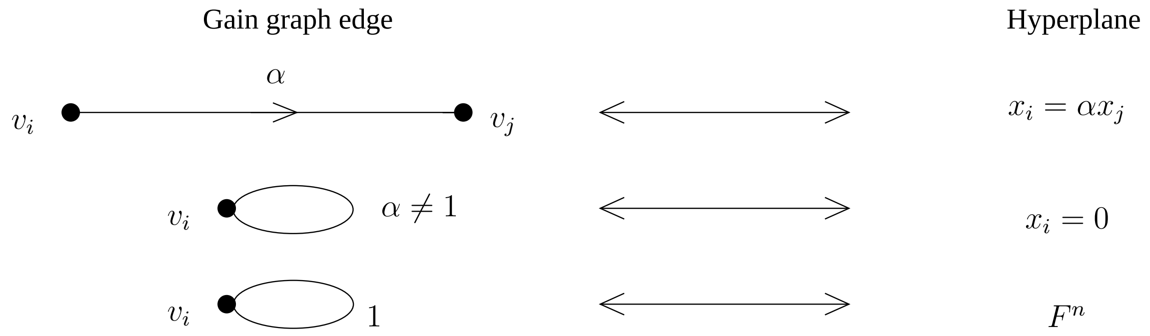
$$p_{(D_m(\mathcal{A}))^\bullet}(\lambda) = m^n p_{\mathcal{A}^\bullet}\left(\frac{\lambda-1}{m}+1\right), \tag{1.1}$$

where $\mathcal{A}^\bullet$ is the hyperplane arrangement consisting of the members of $\mathcal{A}$ together with the coordinate hyperplanes not already in $\mathcal{A}$.

A *frame arrangement* is a hyperplane arrangement consisting of hyperplanes of the form $x_i = \alpha x_j$ ($i \neq j$, $\alpha \neq 0$) or $x_i = 0$. Let $\mathcal{A}$ be a positive real frame arrangement, and let ζ be a primitive mth root of unity. The *Dowlingization* $D_m(\mathcal{A})$ consists of the complex hyperplanes $x_i = \zeta^h \alpha x_j$, $0 \leq h \leq m-1$, together with any coordinate hyperplanes in $\mathcal{A}$. Ehrenborg and Readdy mention that both arrangements can be encoded as gain graphs (see Figure **??**). However, they do not employ the theory of gain and biased graphs in their proofs.

Armed with the theory of gain and biased graphs, we extend (**??**). First we translate this equation to one about gain graphs. Let the gain graph Φ encode $\mathcal{A}$, and let $\langle\zeta\rangle\Phi$ encode $D_m(\mathcal{A})$. (Here $\langle\zeta\rangle\Phi$ is just the name of a gain graph, but I chose this name carefully.) We

FIGURE 1.1. This is a description of how a gain graph Φ encodes the hyperplanes of a frame arrangement $\mathcal{H}(\Phi)$. The gains are elements of a multiplicative subgroup of a skew field F, and Φ has n vertices.



explain in Section **??** (see (**??**)) that the characteristic polynomial of a frame arrangement equals the *chromatic polynomial* $\chi_\Phi(\lambda)$ of the associated gain graph Φ. Thus we can rewrite (**??**) as

$$\chi_{(\langle\zeta\rangle\Phi)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda-1}{m}+1\right). \tag{1.2}$$

Let $\mathfrak{G}$ be a group with m elements, and let Δ be an ordinary (not biased) graph. Zaslavsky proved results in [**?**] that, when combined, yield the following formula:

$$\chi_{(\mathfrak{G}\Delta)^\bullet}(\lambda) = m^n \chi_{\langle\Delta\rangle^\bullet}\left(\frac{\lambda-1}{m}+1\right). \tag{1.3}$$

The resemblance between (**??**) and (**??**) is not coincidental.

Corollary **??**, a consequence of our main result, contains both (**??**) and (**??**) as special cases. The corollary includes a formula like those above for a *group expansion of a gain graph*, a concept we introduce. Equation (**??**) is the case in which a gain graph with positive real gains is expanded by the group generated by a primitive root of unity. In (**??**), a balanced biased graph is expanded by a finite group.

The main result in this paper is Theorem **??**, a biased-graph generalization of Corollary **??**. This theorem is about the chromatic polynomial of a *biased expansion of a biased graph*.

This new type of biased graph is an abstraction of a group expansion of a gain graph. The inspiration for this level of generality is Zaslavsky's work on biased expansions of ordinary graphs (first defined in [?, Example 3.8], with significant development in [?] and [?]).

We employ several aspects of biased graph theory to prove Theorem ??. For certain expansions of gain graphs, we use gain-graph coloring, a concept introduced in [?, Section 4]. The general proof incorporates algebraic results from [?]. Zaslavsky used both techniques to prove the results that imply (??).

We switch gears a bit at the end of this paper. A common question to ask about a characteristic polynomial is: When are the roots integral? Supersolvability is a matroid property that implies that the roots are integral. We describe which biased expansions of biased graphs have supersolvable biased matroids.

2. Gain and Biased Graphs

Zaslavsky is developing a theory of gain and biased graphs in a series of papers entitled "Biased Graphs". Here we extract definitions that are necessary for this paper. Let $\Gamma = (V, E)$ be a graph. In this paper, all vertex and edge sets are finite. Edges are *links* (two distinct endpoints) or *loops* (two coincident endpoints).

A *circle* is the edge set of a closed path. A *theta graph* is the union of three internally disjoint open paths with the same two endpoints. A *tight bracelet* is the union of two circles that intersect at precisely one vertex. A *loose handcuff* is the union of two vertex-disjoint circles and a path meeting each circle at one endpoint and nowhere else. These graphs are depicted in Figure ??. By $m\Gamma$ we mean a graph Γ with every edge replaced by m copies of itself. A *star* S_k is a simple graph having k edges that are all incident to one vertex.

An *induced subgraph* of Γ is $\Gamma{:}W = (W, E{:}W)$ where $W \subseteq V$ and $E{:}W$ consists of the edges of E whose vertices are contained in W. W is *stable* if $E{:}W = \emptyset$. An *edge-induced*

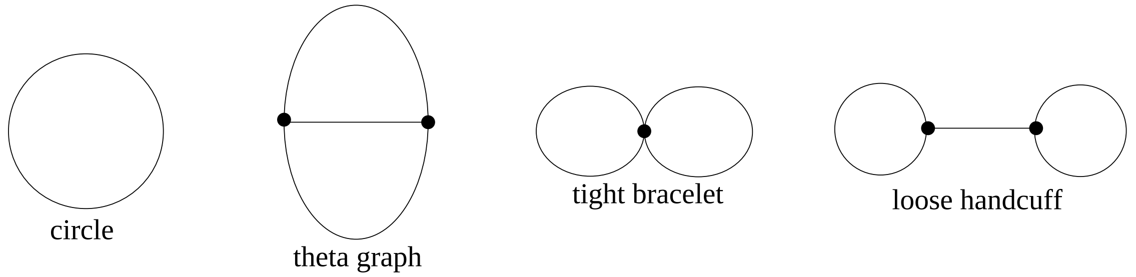


FIGURE 2.1. These graphs are the underlying graphs of the circuits of the bias matroid.

subgraph of Γ is $\Gamma{:}S = \big(V(S), S\big)$ where $S \subseteq E$ and $V(S)$ consists of those vertices that are incident with an edge in S. We let $c(S)$ equal the number of connected components of (V, S). We write $c(\Gamma)$ instead of $c(E)$.

A *biased graph* $\Omega = (\Gamma, \mathcal{B})$ consists of a graph Γ and a subclass $\mathcal{B}$ of the class of circles of Γ which satisfies the property: if two circles of a theta graph are in $\mathcal{B}$, then the third circle is also in $\mathcal{B}$. Such a subclass is called a *linear class of circles*. We call $\mathcal{B}$ the set of *balanced circles* of Ω. If writing only $\mathcal{B}$ is ambiguous, we write $\mathcal{B}(\Omega)$ instead. We call Γ the *underlying graph* of Ω. If Γ is not explicitly defined, we write $||\Omega||$ instead. We write $E(\Omega) = E$ and $V(\Omega) = V$.

A *homomorphism* (or *map*) of graphs is an incidence-preserving mapping of the vertex and edge sets. That is, a homomorphism $f : \Gamma \to \Gamma'$ consists of mappings $f_V : V \to V'$ and $f_E : E \to E'$ such that $f_V(v)$ and $f_E(e)$ are incident in Γ' if $v \in V$ and $e \in E$ are incident in Γ. A homomorphism is an *isomorphism* if f_V and f_E are bijections and if $v \in V$ and $e \in E$ are incident in Γ if and only if $f_V(v)$ and $f_E(e)$ are incident in Γ'. Two biased graphs Ω_1 and Ω_2 are *isomorphic* when there is an isomorphism $f : ||\Omega_1|| \to ||\Omega_2||$ such that $\mathcal{B}(\Omega_2) = \big\{ f_E(C_1) \mid C_1 \in \mathcal{B}(\Omega_1) \big\}$.

In a biased graph Ω, we let $U(\Omega) = \{v \in V \mid v \text{ supports an unbalanced loop}\}$. The biased graph $\Omega^\bullet$ denotes Ω with an unbalanced loop added at each vertex in $V \backslash U(\Omega)$. We call Ω *simply biased* if it has no balanced loops, balanced digons, or pairs of unbalanced loops at the same vertex.

If $W \subseteq V$ and $S \subseteq E$, then $\Omega{:}W$ and $\Omega{:}S$ denote the respective subgraphs $\Gamma{:}W$ and $\Gamma{:}S$ with balance of circles the same as in Ω.

The edge set S is *balanced* if every circle in it is balanced and is *contrabalanced* if it contains no balanced circle. We say that Ω is balanced or contrabalanced if its edge set is balanced or contrabalanced, respectively.

We let $b(S)$ equal the number of connected components of (V, S) which are balanced. We write $b(\Omega)$ instead of $b\big(E(\Omega)\big)$. The *balance-closure* of S, denoted $\operatorname{bcl}(S)$, is the set

$$S \cup \big\{e \in S^c \mid \text{there is a balanced circle } C \text{ such that } e \in C \subseteq S \cup \{e\}\big\}.$$

Biased graphs are a combinatorial generalization of *gain graphs.* A gain graph $\Phi = (\Gamma, \phi)$ consists of a graph Γ and a *gain mapping* ϕ from the edges of Γ into a group $\mathfrak{G}$, called the *gain group.* We require that $\phi(e^{-1}) = \phi(e)^{-1}$, where e^{-1} means e with its orientation reversed. Thus $\phi(e)$ depends on the orientation of e, but neither orientation is preferred.

Associated with Φ is a class $\mathcal{B}(\Phi)$ of balanced circles. Let B be a circle of Γ. To decide whether or not B is balanced, choose an edge e_1 of B and a direction (clockwise or counterclockwise) to traverse B. Let e_1, e_2, $\ldots$, e_k be the edges of B in the order in which they are traversed, and let them be oriented in this direction. The *gain* of B is $\phi(B) = \phi(e_1)\phi(e_2)\cdots\phi(e_k)$. Then $B \in \mathcal{B}(\Phi)$ if $\phi(B) = 1$. Whether or not $B \in \mathcal{B}(\Phi)$ is independent of the choices of e_1 and the direction in which B is traversed.

A gain graph $\Phi = (\Gamma, \phi)$ yields the biased graph $\big(\Gamma, \mathcal{B}(\Phi)\big)$. We denote this biased graph by $\langle\Phi\rangle$. A biased graph is called *gain biased* if it equals $\langle\Phi\rangle$ for some gain graph.

Let $\mathfrak{G}$ be a group. By $\mathfrak{G}\Gamma$ we mean the gain graph derived from Γ by replacing each edge of Γ by $\#\mathfrak{G}$ new edges, one bearing each possible gain value. We call $\mathfrak{G}\Gamma$ the $\mathfrak{G}$-*expansion* of Γ. We write $\pm\Gamma$ for the sign-group expansion of Γ.

A combinatorial generalization of a group expansion of a graph is a biased expansion. A *biased expansion* of Γ is a biased graph Ω together with a *projection* mapping $\pi : ||\Omega|| \to \Gamma$ that is the identity on vertices, is surjective, and has the *Circle Lifting Property*: for each circle $C = e_1 e_2 \cdots e_l$ in Γ and each $\tilde{e}_1 \in \pi^{-1}(e_1), \ldots, \tilde{e}_{l-1} \in \pi^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in \pi^{-1}(e_l)$ for which $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_l$ is balanced. In addition, for each $e \in E$, all digons in $\pi^{-1}(e)$ are unbalanced.

Associated with a biased graph $\Omega = (\Gamma, \mathcal{B})$ is the *bias matroid* $G(\Omega)$. The point set of $G(\Omega)$ is $E(\Omega)$; and $G(\Omega)$ is easily defined in terms of its circuits, which are the balanced circles and all contrabalanced theta graphs, tight bracelets, and loose handcuffs.

3. A Mini-Course on the Characteristic and Chromatic Polynomials

In this section we give various results about the characteristic and chromatic polynomials. See [**?**] for information about the characteristic polynomial of a geometric lattice, [**?**] for the development of the characteristic and chromatic polynomials that are related to biased graphs, and [**?**] for an introduction to hyperplane arrangements.

The *characteristic polynomial* of a geometric lattice L is defined to be

$$p_L(\lambda) = \sum_{x \in L} \mu_L(\hat{0}, x) \lambda^{r(L) - r(x)},$$

where μ is the Möbius function of L [**?**, Section 7.1], and r is the rank function of L.

Let $\mathcal{A}$ be a hyperplane arrangement in an n-dimensional vector space, and let L be its intersection lattice (ordered by reverse inclusion). The *characteristic polynomial* of $\mathcal{A}$ is defined as

$$p_{\mathcal{A}}(\lambda) = \sum_{x \in L} \mu_L(\hat{0}, x) \lambda^{\dim(x)}$$

if $\mathcal{A}$ does not include the degenerate hyperplane. Otherwise, $p_{\mathcal{A}}(\lambda) = 0$. In general, $p_{\mathcal{A}}(\lambda)$ and $p_L(\lambda)$ need not agree. However, if the degenerate hyperplane is not in $\mathcal{A}$, then

$$p_{\mathcal{A}}(\lambda) = \lambda^{n-r(L)} p_L(\lambda). \tag{3.1}$$

Let M be a matroid whose lattice of flats is L. To define the *characteristic polynomial* of M, we follow [?] and say that

$$p_M(\lambda) = \begin{cases} p_L(\lambda) & \text{if } M \text{ has no loops, and} \\ 0 & \text{otherwise.} \end{cases}$$

Let Γ be a graph, and let $\chi_\Gamma(\lambda)$ be its chromatic polynomial. Then

$$\chi_\Gamma(\lambda) = \lambda^{c(\Gamma)} p_{G(\Gamma)}(\lambda)$$

[?, Proposition 7.5.1], where $G(\Gamma)$ is the matroid of Γ. After applying the Boolean expansion formula for $p_{G(\Gamma)}$ [?, Proposition 7.2.1], we find that

$$\chi_\Gamma(\lambda) = \sum_{S \subseteq E(\Gamma)} \lambda^{c(S)} (-1)^{|S|}.$$

Similar formulas hold for the bias matroid of a biased graph Ω. In [?, Section 3], the chromatic polynomial of Ω is defined as

$$\chi_\Omega(\lambda) = \sum_{S \subseteq E(\Omega)} \lambda^{b(S)} (-1)^{|S|}.$$

Then [?, Theorem 5.1] says that

$$\chi_\Omega(\lambda) = \lambda^{b(\Omega)} p_{G(\Omega)}(\lambda). \tag{3.2}$$

Consider a gain graph Φ whose gain group is a multiplicative subgroup of a skew field, and let Φ have n vertices. Construct $\mathcal{H}(\Phi)$ as described in Figure ??. According to [?, Corollary

2.2], the lattice of flats of the bias matroid $G(\Phi)$, call it L, is isomorphic to the intersection lattice of $\mathcal{H}(\Phi)$ (ordered by reverse inclusion). From (**??**),

$$p_{\mathcal{H}(\Phi)}(\lambda) = \lambda^{n-r(G(\Phi))} p_L(\lambda);$$

and from (**??**),

$$\chi_\Phi(\lambda) = \lambda^{b(\Phi)} p_{G(\Phi)}(\lambda).$$

$\big($For notational ease, we write Φ instead of $\langle\Phi\rangle$.$\big)$ According to [**?**, Theorem 2.1(j)], $b(\Phi) = n - r\big(G(\Phi)\big)$. If Φ contains a balanced loop (which means that $\mathcal{H}(\Phi)$ contains the degenerate hyperplane), then $\chi_\Phi(\lambda) = p_{\mathcal{H}(\Phi)}(\lambda) = 0$. Otherwise, $G(\Phi)$ has no loops, so $p_L(\lambda) = p_{G(\Phi)}(\lambda)$. Thus

$$\chi_\Phi(\lambda) = p_{\mathcal{H}(\Phi)}(\lambda). \tag{3.3}$$

A balanced counterpart of $\chi_\Omega(\lambda)$ is the *balanced chromatic polynomial*

$$\chi^b_\Omega(\lambda) = \sum_{\substack{S \subseteq E(\Omega) \\ S \text{ balanced}}} \lambda^{b(S)} (-1)^{|S|}.$$

As with the chromatic polynomial of a graph, the chromatic polynomials of a gain graph have a coloring interpretation. We briefly describe how to color a gain graph Φ that has a finite gain group $\mathfrak{G}$. For more details, see [**?**, Section 4]. To color Φ in k colors, we assign to each vertex an element of the color set

$$C_k = \big(\{1, \ldots, k\} \times \mathfrak{G}\big) \cup \{0\}.$$

If 0 is never used, the coloring is *zero-free.* An edge is *improper* if it has the form of an edge in Figure **??**. A coloring is *proper* if it has no improper edges. The important theorem about gain-graph coloring is the following:

Theorem 3.1 ([?, Theorem 4.2])**.** *Let Φ be a gain graph, and let k be a nonnegative integer. Then the number of proper colorings of Φ in k colors equals $\chi_\Phi(|\mathfrak{G}|k+1)$ and the number of zero-free proper colorings equals $\chi^b_\Phi(|\mathfrak{G}|k)$.*

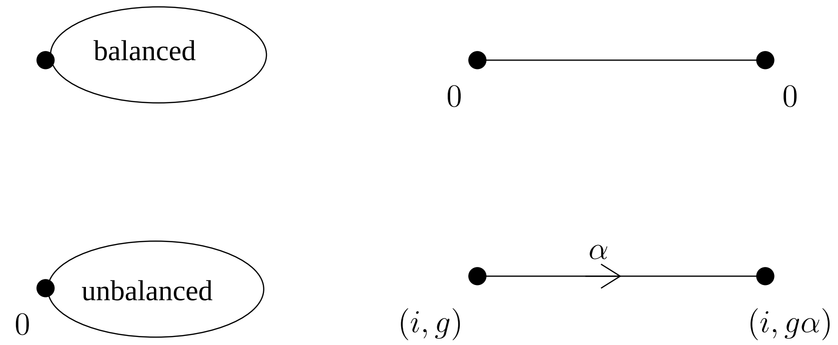


FIGURE 3.1. These are the types of improper edges in gain-graph coloring. The label at a vertex represents its color.

4. BIASED EXPANSIONS OF BIASED GRAPHS

4.1. **Group Expansions of Gain Graphs and the Chromatic Polynomial.** Let Φ be a gain graph with gain group $\mathfrak{H}$, and let $\mathfrak{G}$ be a group. A $\mathfrak{G}$-*expansion* of Φ, denoted by $\mathfrak{G}\Phi$, is a gain graph with gain group $\mathfrak{G} \times \mathfrak{H}$ that is derived from Φ by replacing each edge of Φ with gain h by $\#\mathfrak{G}$ edges, one bearing each possible gain value (g_i, h) for $g_i \in \mathfrak{G}$.

Corollary 4.1. *Let Φ be a gain graph with n vertices, and let $\mathfrak{G}$ be a group with m elements. Then*

$$\chi^b_{\mathfrak{G}\Phi}(\lambda) = m^n \chi^b_\Phi\left(\frac{\lambda}{m}\right). \tag{4.1}$$

Consequently,

$$\chi_{(\mathfrak{G}\Phi)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda-1}{m}+1\right). \tag{4.2}$$

As explained in Section **??**, Ehrenborg and Readdy [?, Theorem 3.2] proved a hyperplane analogue of (**??**) when Φ has gains in $\mathbb{R}^*_{>0}$, the multiplicative group of the positive real numbers, and $\mathfrak{G}$ is the group generated by a primitive mth root of unity. When the gain group of Φ is trivial, Zaslavsky [?, Examples 3.6 and 4.6] proved (??) explicitly and (??) implicitly.

Corollary ?? is actually a special case of Theorem ??, its biased-graph generalization. The proof of Theorem ?? is algebraic. To illustrate gain-graph coloring, however, we first provide a combinatorial proof of (??) when the gain group of Φ is finite. To prove his special case of (??), Zaslavsky used both the algebraic and coloring techniques.

Proof of (??) when Φ has a finite gain group. Throughout this proof we apply Theorem ??. If Φ has a balanced loop, then

$$\chi^b_{\mathfrak{G}\Phi}(\lambda) = m^n \chi^b_{\Phi}\left(\frac{\lambda}{m}\right) = 0.$$

Recall that an unbalanced loop is an improper edge only if its vertex is colored 0. Since this is not a possibility in a zero-free proper coloring, we may assume that Φ has no loops.

Assume the gain group of Φ has p elements. Set $\lambda = pk$, and count zero-free proper colorings of $\mathfrak{G}\Phi$ in k colors. Assume that vertices v_1 and v_2 are adjacent in $\mathfrak{G}\Phi$, and color them $\big(k_1, (q_1, h_1)\big)$ and $\big(k_2, (q_2, h_2)\big)$, respectively. Consult Figure ??. Since $q_1 = gq_2$ for some $g \in \mathfrak{G}$, a proper coloring of $\mathfrak{G}\Phi$ is achieved only if $k_1 \neq k_2$ or if $k_1 = k_2$ and $h_2 \neq h_1 h$. But these are the rules for properly coloring the corresponding vertices of Φ in k colors. To make a zero-free proper coloring of $\mathfrak{G}\Phi$ in k colors, therefore, simply make a zero-free proper coloring of Φ in k colors. Then, for each of the n vertices, arbitrarily choose one of the m values for the $\mathfrak{G}$-coordinate of the color. In other words,

$$\chi^b_{\mathfrak{G}\Phi}(mpk) = m^n \chi^b_{\Phi}(pk).$$

Since this is a polynomial equation valid for all $k \in \mathbb{Z}_{>0}$, it is an identity. $\square$

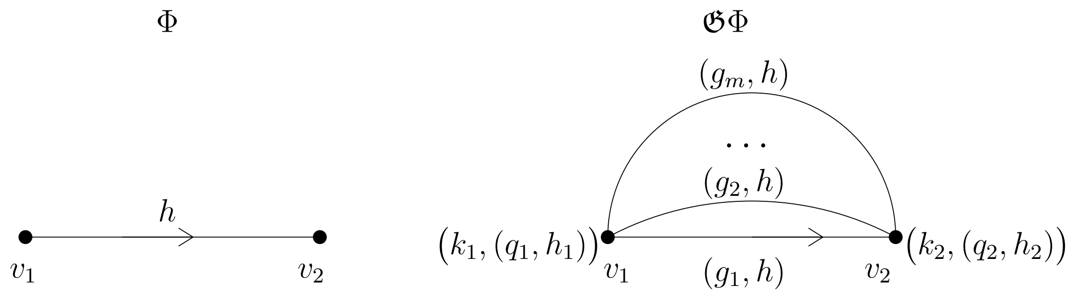


FIGURE 4.1. An edge in Φ and its corresponding edges in $\mathfrak{G}\Phi$.

Recall that the equations of the hyperplanes in a frame arrangement have one of the forms that are shown in Figure **??**.

Corollary 4.2 ([**?**, Theorem 3.2])**.** *Let $\mathcal{A}$ be a frame arrangement in $\mathbb{R}^n$ where the coefficients of the hyperplane equations are positive real numbers. Then*

$$p_{(D_m(\mathcal{A}))^\bullet}(\lambda) = m^n p_{\mathcal{A}^\bullet}\left(\frac{\lambda - 1}{m} + 1\right). \tag{4.3}$$

Proof. Let Φ be the gain graph that encodes $\mathcal{A}$. Let ζ be a primitive mth root of unity, and let Φ_m be the gain graph that encodes $D_m(\mathcal{A})$. So an edge in Φ with gain α is associated with m edges in Φ_m with gains α, $\zeta\alpha$, $\dots$, $\zeta^{m-1}\alpha$.

Since $\mathcal{A}$ does not contain the degenerate hyperplane, neither Φ nor Φ_m contain a balanced loop. If a vertex supports exactly one unbalanced loop, then omitting it does not affect that vertex in $\Phi^\bullet$. If a vertex supports more than one unbalanced loop, then $\mathcal{A}$ contains duplicate coordinate hyperplanes. This does not affect the characteristic polynomial. So we may assume that $\mathcal{A}$ and $D_m(\mathcal{A})$ are loopless.

By (**??**), we need to show that

$$\chi_{(\Phi_m)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda - 1}{m} + 1\right).$$

According to (??),

$$\chi_{(\langle\zeta\rangle\Phi)^\bullet}(\lambda) = m^n \chi_{\Phi^\bullet}\left(\frac{\lambda-1}{m}+1\right).$$

So we only need to show that

$$\chi_{(\Phi_m)^\bullet}(\lambda) = \chi_{(\langle\zeta\rangle\Phi)^\bullet}(\lambda).$$

This is true if Φ_m and $\langle\zeta\rangle\Phi$ are isomorphic. The graph isomorphism is easy to find: it is the identity on vertices, and it maps the edge of $\langle\zeta\rangle\Phi$ with gain (ζ^h, α) to the edge of Φ_m that has gain $\zeta^h\alpha$. Let

$$\zeta^{h_1}\alpha_1, \ldots, \zeta^{h_k}\alpha_k$$

be the gains of the edges of a circle C in Φ_m. This circle is balanced if and only if

$$\zeta^{h_1+\cdots+h_k}\alpha_1\alpha_2\cdots\alpha_k = 1.$$

But $\alpha_i > 0$, so the circle is balanced if and only if

$$\zeta^{h_1+\cdots+h_k} = \alpha_1\alpha_2\cdots\alpha_k = 1.$$

Thus C is balanced if and only if

$$(\zeta^{h_1}, \alpha_1), \ldots, (\zeta^{h_k}, \alpha_k)$$

are the gains of the edges of a balanced circle in $\langle\zeta\rangle\Phi$. □

4.2. **Biased Expansions of Biased Graphs and the Chromatic Polynomial.** Biased graphs arose as a combinatorial generalization of gain graphs. So we ask, what is the biased-graph generalization of a group expansion of a gain graph?

A *biased expansion of a biased graph* Ω is a biased graph Λ together with a *projection* mapping $\pi : ||\Lambda|| \to ||\Omega||$ satisfying:

(BE1) π is the identity on vertices, and $\pi^{-1}(e) \neq \emptyset$ for each $e \in E(\Omega)$;

(BE2) (Balanced Circle Lifting Property) for each balanced circle $C = e_1 e_2 \cdots e_l$ in $\mathcal{B}(\Omega)$ and each $\tilde{e}_1 \in \pi^{-1}(e_1), \ldots, \tilde{e}_{l-1} \in \pi^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in \pi^{-1}(e_l)$ for which $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_l$ is in $\mathcal{B}(\Lambda)$; and

(BE3) (Balanced Circle Projection Property) if a circle $\tilde{C} \in \mathcal{B}(\Lambda)$, then $\pi(\tilde{C}) \in \mathcal{B}(\Omega)$.

A consequence of the Balanced Circle Projection Property is:

(BE4) no balanced digon of Λ projects to a single edge of Ω.

A *lift* of $S \subseteq E(\Omega)$ is any $\tilde{S} \subseteq E(\Lambda)$ such that $\pi(\tilde{S}) = S$ and $\pi|_{\tilde{S}}$ is injective. We will continue to use the notation $\tilde{S}$ for a lift of S.

The *fiber* of an edge e is the set $\pi^{-1}(e)$. An *m-fold* biased expansion of a biased graph Ω is a biased expansion where each edge fiber has m elements. Such an expansion is *regular* and has *multiplicity* m. We denote it by $m \cdot \Omega$.

I chose the definition of a biased expansion of a biased graph because it enables a combinatorial generalization of Corollary **??**. Also, it generalizes Zaslavsky's definition of a biased expansion of a graph.

Notice that $\langle \mathfrak{G}\Phi \rangle$ is a $|\mathfrak{G}|$-fold biased expansion of the biased graph $\langle \Phi \rangle$.

Here is the generalization of Corollary **??**.

Theorem 4.3. *Let Λ be an m-fold biased expansion of Ω, and assume that Ω has n vertices. Then*

$$\chi^b_{\Lambda}(\lambda) = m^n \chi^b_{\Omega}\left(\frac{\lambda}{m}\right). \tag{4.4}$$

Consequently,

$$\chi_{\Lambda^\bullet}(\lambda) = m^n \chi_{\Omega^\bullet}\left(\frac{\lambda - 1}{m} + 1\right). \tag{4.5}$$

Proof. We begin with a proof of (**??**). This proof closely mimics Zaslavsky's proof for group expansions of graphs [**?**, Example 3.6]. Consider the following string of equalities:

$$\sum_{\substack{S\subseteq E(\Lambda)\\ S \text{ balanced}}} \lambda^{b(S)}(-1)^{|S|} = \sum_{\substack{S\subseteq E(\Lambda)\\ S \text{ balanced}}} \lambda^{b(\pi(S))}(-1)^{|\pi(S)|} = \sum_{\substack{T\subseteq E(\Omega)\\ T \text{ balanced}}} m^{n-b(T)}\lambda^{b(T)}(-1)^{|T|}.$$

By definition, the first expression is $\chi^b_\Lambda(\lambda)$ and the last expression is $m^n\chi^b_\Omega\left(\frac{\lambda}{m}\right)$. To prove the first equality, we show that $\Lambda{:}S$ and $\Omega{:}\pi(S)$ are isomorphic. That π is an isomorphism is clear, except for showing that the edge-set mapping is injective. Let e_1 and e_2 be in S. If $\pi(e_1) = \pi(e_2)$, then e_1e_2 is a digon in Λ. By (BE4), e_1e_2 is unbalanced, a contradiction. Thus $|S| = |\pi(S)|$, and S and $\pi(S)$ have the same number of components. The first equality is a consequence of these facts, together with the fact that $\pi(S)$ is balanced, which follows from the Balanced Circle Projection Property.

For the second equality to hold, we must show that a balanced subset T of $E(\Omega)$ has $m^{n-b(T)}$ balanced lifts. Specify a maximal forest F of T. Each balanced lift of T includes a lift of F. Lift F to $\tilde{F}$. This can be done in $m^{n-b(T)}$ ways. By the Balanced Circle Lifting Property (BE2), there is at most one balanced lift of T containing $\tilde{F}$. Since $\tilde{F}$ is balanced, so is $\mathrm{bcl}(\tilde{F})$ [**?**, Proposition 3.1]. So $\mathrm{bcl}(\tilde{F}) \cap \pi^{-1}(T)$ is such a lift. Thus T has $m^{n-b(T)}$ balanced lifts.

Now we prove Equation (**??**). Since unbalanced loops do not affect the balanced chromatic polynomial,

$$\chi^b_{\Lambda^\bullet}(\lambda) = m^n\chi^b_{\Omega^\bullet}\left(\frac{\lambda}{m}\right). \tag{4.6}$$

If each vertex in a biased graph Ω supports an unbalanced loop, then $\chi_\Omega(\lambda) = \chi^b_\Omega(\lambda - 1)$ [**?**, Equation 11.1]. Thus,

$$\chi_{\Lambda^\bullet}(\lambda + 1) = m^n\chi_{\Omega^\bullet}\left(\frac{\lambda}{m} + 1\right).$$

$\square$

5. Supersolvability

A question that is commonly asked about the characteristic polynomial of a matroid is: When are the roots integral? In [?], Stanley introduced a matroid property called supersolvability that provides a partial answer. A matroid is *supersolvable* if it has a complete chain of modular flats. Stanley showed that supersolvable matroids have roots that are positive integers [?, Theorem 4.1]. For a graph Γ, he also showed that the graphic matroid $G(\Gamma)$ is supersolvable if and only if Γ is chordal [?, Proposition 2.8]. In [?], Zaslavsky proved the following theorem, that generalizes Stanley's result by classifying the biased graphs that have supersolvable bias matroids. The proof in [?] is incomplete, but I corrected it in [?].

Theorem 5.1 ([?, Theorem 2.2])**.** *Let Ω be a simply biased graph. $G(\Omega)$ is supersolvable if and only if each connected component of Ω either:*

(1) *has a bias-simplicial vertex ordering; or*
(2) *is a simplicial extension of one of*
   (a) $(mK_2, \emptyset)$ *where $m \geq 2$, or*
   (b) $\langle \pm K_3 \rangle$*, or*
   (c) $\langle +\Gamma \cup -S_k \rangle$*, where Γ is a chordal simple graph, S_k is a k-edge star whose vertex set lies in $V(\Gamma)$, and the noncentral vertices of S_k are a clique in Γ.*

Before we proceed, we define a few concepts from [?] that appear either in Theorem **??** or later in this section. In a biased graph Ω, a vertex v is *bias simplicial* if:

(s1) for each pair of edges, e and f, from v to distinct neighbors x and y, there is an xy edge which completes a balanced triangle;
(s2) for each unbalanced digon that has one endpoint at v, the other endpoint is in $U(\Omega)$; and
(s3) if v is in $U(\Omega)$, then every neighbor is in $U(\Omega)$.

We call v *simplicial* if it satisfies (s1), is not in $U(\Omega)$, and is not in an unbalanced digon. A *bias-simplicial vertex ordering* (b.s.v.o.) of Ω is a linear ordering of the vertices, say $(v_1, \ldots, v_n)$, such that each v_i is bias simplicial in $\Omega{:}\{v_1, \ldots, v_i\}$. We call Ω_0 a *simplicial extension* of Ω if Ω is an induced subgraph of Ω_0 and $V(\Omega)^c$ can be linearly ordered, say $(w_1, \ldots, w_n)$, so that each w_i is simplicial in $\Omega_0{:}\,(V(\Omega) \cup \{w_1, \ldots, w_i\})$. We call Ω a *base* of Ω_0.

Theorem **??** characterizes the biased expansions that have supersolvable bias matroids. Certainly Theorem **??** is instrumental in the proof. Notice, however, that it applies to simply biased graphs. This is slightly annoying because a biased expansion Λ of a simply biased graph Ω need not be simply biased. The problem occurs when v supports an unbalanced loop in Ω. Then it may support multiple unbalanced loops in Λ. In the proof of Lemma **??**, we fix the problem by letting Λ' be Λ less all but one unbalanced loop at each vertex. (If a vertex does not support an unbalanced loop in Λ, then Λ' has no unbalanced loop there). So if Λ' has no unbalanced loop, then $\Lambda' = \Lambda$. Clearly, $G(\Lambda)$ is supersolvable if and only if $G(\Lambda')$ is supersolvable. Also, Λ has a b.s.v.o. if and only if Λ' has a b.s.v.o. (the same ordering works).

In the proofs of our supersolvability results, we assume that the biased graphs are connected. We can do this because $G(\Omega)$ is supersolvable if and only if $G(\Omega')$ is supersolvable for each component Ω' of Ω. The proof of this is straight-forward given the fact that if H is a modular copoint of $G(\Omega)$, then $E(\Omega') \cap H$ is either $E(\Omega')$ or a modular copoint of $G(\Omega')$.

Lemma 5.2. *Let Λ be a biased expansion of a simply biased graph Ω. Assume that $|\pi^{-1}(e)| \geq 2$ for all $e \in E(\Omega)$. $G(\Lambda)$ is supersolvable if and only if each component of Λ either:*

(1) *has a bias-simplicial vertex ordering, or*
(2) *is $\langle \pm K_3 \rangle$ or $(mK_2, \emptyset)$ for some $m \geq 2$.*

Proof. Sufficiency follows immediately from Theorem **??**. We may assume that Λ is connected. Assume that $G(\Lambda)$ is supersolvable but does not have a b.s.v.o. Then Λ' is a simplicial extension of one of the base graphs in Theorem **??** (**??**). A simplicial vertex not in the base cannot be contained in an unbalanced digon. Since each link in Λ' is in an unbalanced digon, Λ' must actually be one of the base graphs. If $\langle +\Gamma \cup -S_k \rangle$ has each link in an unbalanced digon, then $\langle +\Gamma \cup -S_k \rangle = (2K_2, \emptyset)$. Thus Λ', and hence Λ, has the specified form. □

Lemma 5.3. *Let Λ be a biased expansion of a simply biased graph Ω. Assume that $|\pi^{-1}(e)| \geq 2$ for all $e \in E(\Omega)$. If $G(\Lambda)$ is supersolvable, then $G(\Omega)$ is supersolvable.*

Proof. We may assume that Λ is connected. Assume that $(v_1, \ldots, v_n)$ is a b.s.v.o. of Λ. We show that it is also a b.s.v.o. of Ω. Suppose that in $\Omega{:}\{v_1, \ldots, v_i\}$, v_i is incident with links e and f and that they do not form a digon. Choose lifts $\tilde{e}$ and $\tilde{f}$. Then there exists $\tilde{g}$ such that $\tilde{e}\tilde{f}\tilde{g}$ is balanced in $\Lambda{:}\{v_1, \ldots, v_i\}$. By (BG3), efg is a balanced triangle in $\Omega{:}\{v_1, \ldots, v_i\}$.

Suppose that v_i is contained in an unbalanced digon ef in $\Omega{:}\{v_1, \ldots, v_i\}$. Call the other vertex in the digon v_j. Then, by (BG3), any lift $\tilde{e}\tilde{f}$ is unbalanced. Thus $v_j \in U(\Lambda)$. But $U(\Lambda) = U(\Omega)$ because Ω has no balanced loops.

Lastly, suppose that $v_i \in U(\Omega)$ and let v_j be one of its neighbors in $\Omega{:}\{v_1, \ldots, v_i\}$. Then $v_i \in U(\Lambda)$, so $v_j \in U(\Lambda)$. Accordingly, $v_j \in U(\Omega)$.

If Λ does not have a b.s.v.o., then Lemma **??** gives the form of Λ. If $\Lambda = \langle \pm K_3 \rangle$, then $\Omega = \langle K_3 \rangle$; and if $\Lambda = (mK_2, \emptyset)$, then $\Omega = (m_1 K_2, \emptyset)$ where $2m_1 \leq m_2$. In both cases, $G(\Omega)$ is supersolvable. □

The converse of Lemma **??** is not true. Let Ω consist of a balanced triangle with an unbalanced loop at one vertex. Let $\Lambda = \langle \mathbb{Z}_2 \Omega \rangle$. Then $G(\Omega)$ is supersolvable, but $G(\Lambda)$ is not. Theorem **??** says that the converse will fail when Ω has adjacent vertices, neither of which supports an unbalanced loop.

Theorem 5.4. *Let Λ be a biased expansion of a simply biased graph Ω. Assume $|\pi^{-1}(e)| \geq 2$ for all $e \in E(\Omega)$. $G(\Lambda)$ is supersolvable if and only if for each component Ω' of Ω either:*

(1) *Ω' has a bias-simplicial vertex ordering and there is no pair of adjacent vertices in $U(\Omega')^c$; or*
(2) *Ω' is $\langle K_3 \rangle$ (and the corresponding component of Λ is $\langle \pm K_3 \rangle$); or*
(3) *Ω' is $(m_1 K_2, \emptyset)$ (and the corresponding component of Λ is $(m_2 K_2, \emptyset)$ with $2m_1 \leq m_2$).*

Proof. We may assume that Ω is connected. We begin with a proof of sufficiency. According to Theorem **??**, (**??**) and (**??**) imply that $G(\Lambda)$ is supersolvable. If (**??**) holds and $(v_1, \ldots, v_n)$ is a b.s.v.o. of Ω, we show that it is also a b.s.v.o. of Λ. In Λ:$\{v_1, \ldots, v_i\}$, assume that v_i is incident with links $\tilde{e}$ and $\tilde{f}$ and that these edges do not form a digon. Since v_i is bias simplicial in Ω, there exists $g \in E(\Omega)$ such that $\pi(\tilde{e})\pi(\tilde{f})g$ is a balanced triangle in Ω. By the Balanced Circle Lifting Property, there exists $\tilde{g}$ such that $\tilde{e}\tilde{f}\tilde{g}$ is a balanced triangle in Λ. Now assume that Λ:$\{v_1, \ldots, v_i\}$ has an unbalanced digon $\tilde{D}$ at v_i. Let v_j be the second vertex of $\tilde{D}$. Since Ω has no balanced digons, $\tilde{D}$ projects to an unbalanced digon or to a link. If $\pi(\tilde{D})$ is an unbalanced digon, then v_j must support an unbalanced loop in Ω. Hence $v_j \in U(\Lambda)$. If $\pi(\tilde{D})$ is a link, then at least one of v_j and v_i is in $U(\Omega)$ by hypothesis. Since v_j precedes v_i in the b.s.v.o. of Ω, $v_j \in U(\Omega)$. But then $v_j \in U(\Lambda)$ too.

Now we prove necessity. Assume Ω has a link e, both of whose vertices are not in $U(\Omega)$. Since $|\pi^{-1}(e)| \geq 2$, Λ cannot have a b.s.v.o. According to Lemma **??**, $\Lambda = \langle \pm K_3 \rangle$ or $\Lambda = (m_2 K_2, \emptyset)$ for some $m_2 \geq 2$. Then Ω is $\langle K_3 \rangle$ or $(m_1 K_2, \emptyset)$ where $2m_1 \leq m_2$, respectively. Finally, assume that Ω does not have a b.s.v.o. Then neither does Λ (see the proof of Lemma **??**), and we just analyzed this possibility. □

A special case and an application of Theorem **??** are [**?**, Theorem 6.4] and [**?**, Theorem 4.1].

Corollary 5.5 ([?, Theorem 6.4]). *Assume that* Γ *is connected and* $|\mathfrak{G}| \geq 2$ *for some group* $\mathfrak{G}$. *Then* $G(\mathfrak{G}\Gamma^{(H)})$ *is supersolvable if and only if* Γ *is chordal and either* Γ *has at most two vertices, or* H^c *is a stable set of simplicial vertices, or* $|\mathfrak{G}| = 2$ *and* $\Gamma = K_3$ *and* $H = \emptyset$.

Corollary 5.6 ([?, Theorem 4.1]). *Let* $\mathcal{A}$ *be a frame arrangement that includes all coordinate hyperplanes. If* $\mathcal{A}$ *is supersolvable, then so is the Dowlingization* $D_m(\mathcal{A})$.

Proof. Let Φ be the gain graph associated with $\mathcal{A}$, and let $\mathfrak{G}\Phi$ be the gain graph associated with its Dowlingization. Then Φ has an unbalanced loop at each vertex. Since the graphs in Theorem **??**(**??**) do not have loops, $\langle\Phi\rangle$ must have a b.s.v.o. By Theorem **??**, $G(\mathfrak{G}\Phi)$ is supersolvable too. □

In [?], the statement of Corollary **??** is incorrect. The result is stated for frame arrangements (which, as defined in [?], need not contain the coordinate hyperplanes). The proof uses the coordinate hyperplanes, however, so the authors intended them to be part of the result.

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