TEST #3

Math 142 **Name:** ______________________________

Problem	1	2	3	4	5	Total
Possible Score	20	80	15	25	60	200
Your Score						

SHOW ALL WORK. Any solution that is not accompanied by the appropriate work necessary for solving the problem will receive no credit. Do not use your calculator to evaluate any limits, derivatives, or integrals. If you need more space, you may use the back of the page.

ERROR APPROXIMATION FOR THE RATIO TEST

If the series is alternating, then use the error for an alternating series. Suppose each $a_n > 0$, let $r_n = \dfrac{a_{n+1}}{a_n}$, and suppose $\lim_{n\to\infty} r_n = L$. Then:

- if $\{r_n\}$ is decreasing, then $R_k \leq \dfrac{a_{k+1}}{1 - r_{k+1}}$.
- if $\{r_n\}$ is increasing, then $R_k \leq \dfrac{a_{k+1}}{1 - L}$.

1. (20 pts) Find the power series for $g(x) = \arctan(3x^2)$ (**Hint:** use the derivative of $\arctan(3x^2)$ and **show all of your work**).

SOLUTION: $\dfrac{d}{dx}\left(\arctan(3x^2)\right) = \dfrac{6x}{1+9x^4}$, so we have

$$\begin{aligned}
\arctan(3x^2) &= \int \frac{6x}{1+9x^4}\,dx \\
&= \int \frac{6x}{1-(-9x^4)}\,dx \\
&= \int \sum_{n=0}^{\infty} 6x(-9x^4)^n\,dx \\
&= \int \sum_{n=0}^{\infty} 6x(-9)^n x^{4n}\,dx \\
&= \int \sum_{n=0}^{\infty} 6(-9)^n x^{4n+1}\,dx \\
&\boxed{= \sum_{n=0}^{\infty} \frac{6(-9)^n}{4n+2}\,x^{4n+2}}
\end{aligned}$$

2. Determine whether the following series converge or diverge. Be sure to indicate which test you are using and **show all of your work**. These problems are on the next four pages.

(a) (20 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^5+3n^4-5n+6}}{n^4-6n^2+7n}$

SOLUTION: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^5+3n^4-5n+6}}{n^4-6n^2+7n} \approx \sum_{n=1}^{\infty} \frac{\sqrt{4n^5}}{n^4} = \sum_{n=1}^{\infty} \frac{2n^{5/2}}{n^4} = 2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$

So we will use the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\sqrt{4n^5+3n^4-5n+6}}{n^4-6n^2+7n}}{\frac{1}{n^{3/2}}} &= \lim_{n\to\infty} \frac{\sqrt{4n^5+3n^4-5n+6}}{n^4-6n^2+7n} \cdot n^{3/2} \\
&= \lim_{n\to\infty} \frac{\sqrt{4n^5}}{n^4} \cdot n^{3/2} \\
&= \lim_{n\to\infty} \frac{2n^{5/2}}{n^4} \cdot n^{3/2} \\
&= \lim_{n\to\infty} \frac{2n^4}{n^4} \\
&= 2 > 0
\end{aligned}
$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (it's a p-series with $p = \frac{3}{2} > 1$), by the limit comparison test, we have $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^5+3n^4-5n+6}}{n^4-6n^2+7n}$ $\boxed{\text{converges}}$

(b) (20 pts) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n+3}{5^n}$

SOLUTION: Since this series is alternating, we use the alternating series test.

$$(1)\ \lim_{n\to\infty}\frac{n+3}{5^n} \overset{LH}{=} \lim_{n\to\infty}\frac{1}{\ln(5)\cdot 5^n} = 0.$$

$$(2)\ \frac{d}{dn}\left(\frac{n+3}{5^n}\right) = \frac{5^n - \ln(5)\cdot 5^n(n+3)}{5^{2n}} = \frac{1-\ln(5)(n+3)}{5^n}$$

$$1-\ln(5)(n+3) < 0 \Rightarrow 1 < \ln(5)(n+3) \Rightarrow \frac{1}{\ln(5)} < n+3 \Rightarrow -2.4 \approx \frac{1}{\ln(5)} - 3 < n$$

Thus, $\displaystyle\frac{d}{dn}\left(\frac{n+3}{5^n}\right) < 0$ for $n \geq 1$, so the sequence is decreasing.

So by the alternating series test, $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n+3}{5^n}$ $\boxed{\text{converges}}$

(c) (20 pts) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(2\cdot 4\cdot 6\cdot\ldots\cdot 2n)}{8\cdot 13\cdot 18\cdot\ldots\cdot(5n+3)}$

SOLUTION: We use the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{3^{n+1}(2\cdot 4\cdot 6\cdot\ldots\cdot 2n\cdot(2n+2))}{8\cdot 13\cdot 18\cdot\ldots\cdot(5n+3)\cdot(5n+8)}\cdot\frac{8\cdot 13\cdot 18\cdot\ldots\cdot(5n+3)}{3^n(2\cdot 4\cdot 6\cdot\ldots\cdot 2n)}\right| &= \lim_{n\to\infty}\frac{3(2n+2)}{5n+8}\\ &= \lim_{n\to\infty}\frac{6n+6}{5n+8}\\ &= \frac{6}{5} > 1\end{aligned}$$

Therefore, by the ratio test, we have $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(2\cdot 4\cdot 6\cdot\ldots\cdot 2n)}{8\cdot 13\cdot 18\cdot\ldots\cdot(5n+3)}$ $\boxed{\text{diverges}}$

(d) (20 pts) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n}{5n-3}\right)^n$

SOLUTION: We use the test for divergence.

$$\lim_{n\to\infty}\left(\frac{5n}{5n-3}\right)^n \to 1^{\infty}, \text{ so let } L = \lim_{n\to\infty}\left(\frac{5n}{5n-3}\right)^n.$$

$$\begin{aligned}
\text{Thus, } \ln(L) &= \lim_{n\to\infty} \ln\left(\left(\frac{5n}{5n-3}\right)^n\right) \\
&= \lim_{n\to\infty} n\cdot\ln\left(\frac{5n}{5n-3}\right) \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{5n}{5n-3}\right)}{n^{-1}} \\
&\overset{LH}{=} \lim_{n\to\infty} \frac{\frac{5n-3}{5n}\cdot\frac{5(5n-3)-5(5n)}{(5n-3)^2}}{-n^{-2}} \\
&= \lim_{n\to\infty} -n^2\cdot\frac{5n-3}{5n}\cdot\frac{-15}{(5n-3)^2} \\
&= \lim_{n\to\infty} \frac{15n^2}{5n(5n-3)} \\
&= \frac{15}{25} = \frac{3}{5}
\end{aligned}$$

Therefore, $\ln L = .6 \Rightarrow L = e^{.6} \neq 0$, so by the test for divergence,

$$\sum_{n=1}^{\infty}\left(\frac{5n}{5n-3}\right)^n \ \boxed{\text{diverges}}$$

3. (15 pts) Find the domain of $h(x) = \sum_{n=0}^{\infty} \frac{n!}{1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n+1)} (x-1)^{2n}$. You do NOT need to check the endpoints of the interval that you find.

SOLUTION: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-1)^{2n+2}}{1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n+1) \cdot (3n+4)} \cdot \frac{1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n+1)}{n!(x-1)^{2n}} \right| = \lim_{n\to\infty} \frac{(x-1)^2(n+1)}{3n+4}$$

$$= \frac{(x-1)^2}{3}$$

$$\frac{(x-1)^2}{3} < 1 \Rightarrow (x-1)^2 < 3 \Rightarrow -\sqrt{3} < x-1 < \sqrt{3} \Rightarrow \boxed{1-\sqrt{3} < x < 1+\sqrt{3}}$$

4. (25 pts) We can use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{n}{n^5+3}$ converges (you do not need to show this, but you should think about what series you would compare it to). How many terms should we sum to approximate the series to with .000001?

SOLUTION: $\sum_{n=1}^{\infty} \frac{n}{n^5+3} \approx \sum_{n=1}^{\infty} \frac{n}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^4}$, so we compare the series to $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Thus, $R_k = \sum_{n=k+1}^{\infty} \frac{n}{n^5+3} \leq \sum_{n=k+1}^{\infty} \frac{1}{n^4}$

$$\leq \int_k^{\infty} x^{-4}\, dx$$

$$= \lim_{t\to\infty} \Big|_k^t -\frac{1}{3x^3}$$

$$= \lim_{t\to\infty} -\frac{1}{3t^3} + \frac{1}{3k^3}$$

$$= \frac{1}{3k^3}$$

So we want $\frac{1}{3k^3} \leq \frac{1}{1000000} \Rightarrow 1000000 \leq 3k^3$

$$\Rightarrow \frac{1000000}{3} \leq k^3$$

$$\Rightarrow 69.3 \approx \sqrt[3]{\frac{1000000}{3}} \leq k$$

Therefore, $\boxed{k = 70}$

5. Let $f(x) = \dfrac{3x^2}{\sqrt[3]{8+2x^2}}$.

(a) (30 pts) Find the power series for $f(x)$. Simplify your answer.

SOLUTION:
$$\frac{3x^2}{\sqrt[3]{8+2x^2}} = \frac{3x^2}{\sqrt[3]{8}\cdot\sqrt[3]{1+\frac{x^2}{4}}}$$
$$= \frac{3}{2}x^2\cdot\left(1+\frac{x^2}{4}\right)^{-1/3}$$
$$= \frac{3}{2}x^2\sum_{n=0}^{\infty}\binom{-1/3}{n}\left(\frac{x^2}{4}\right)^n$$

$$\binom{-1/3}{n} = \frac{\left(-\frac{1}{3}\right)!}{n!\cdot\left(-\frac{1}{3}-n\right)!}$$
$$= \frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)\cdot\ldots\cdot\left(\frac{2}{3}-n\right)\cdot\left(-\frac{1}{3}-n\right)!}{n!\cdot\left(-\frac{1}{3}-n\right)!}$$
$$= \frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)\cdot\ldots\cdot\left(\frac{2}{3}-n\right)}{n!}$$
$$= \frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)\cdot\ldots\cdot\left(-\frac{3n-2}{3}\right)}{n!}$$
$$= \frac{(-1)^n\cdot\left(\frac{1}{3}\right)^n\cdot\left(1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)\right)}{n!}$$

This sequence is for all $n \geq 1$, so for $n = 0$, we get $\binom{-1/3}{0} = 1$.

Therefore,
$$f(x) = \frac{3}{2}x^2\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n\left(1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)\right)}{3^n\cdot n!}\cdot\frac{x^{2n}}{4^n}\right)$$

$$\boxed{= \frac{3}{2}x^2+\sum_{n=1}^{\infty}\frac{3\cdot(-1)^n\left(1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)\right)}{2\cdot 12^n\cdot n!}\cdot x^{2n+2}}$$

(b) (15 pts) Use your answer from part (a) to find $\int_0^2 f(x)\,dx$. Your answer should be in the form of a simplified series.

SOLUTION: $$\int_0^2 f(x)\,dx = \int_0^2 \frac{3}{2}x^2 + \sum_{n=1}^{\infty} \frac{3\cdot(-1)^n\big(1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)\big)}{2\cdot 12^n\cdot n!}\cdot x^{2n+2}\,dx$$

$$= \Big|_0^2 \frac{1}{2}x^3 + \sum_{n=1}^{\infty} \frac{3\cdot(-1)^n\big(1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)\big)}{2\cdot 12^n\cdot n!\cdot(2n+3)}\cdot x^{2n+3}$$

$$\boxed{= 4 + \sum_{n=1}^{\infty} \frac{3\cdot(-1)^n\big(1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)\big)\cdot 2^{2n+3}}{2\cdot 12^n\cdot n!\cdot(2n+3)}}$$

(c) (15 pts) Using your answer from part (b), what is the error in using S_4 to approximate $\int_0^2 f(x)\,dx$?

SOLUTION: Since the answer from part (b) is an alternating series, we have

$$|R_4| \le a_5$$

$$= \frac{3\cdot\big(1\cdot 4\cdot 7\cdot 10\cdot 13\big)\cdot 2^{13}}{2\cdot 12^5\cdot 5!\cdot 13}$$

$$\boxed{.115}$$