TEST #2

Math 142 **Name:** ______________________________

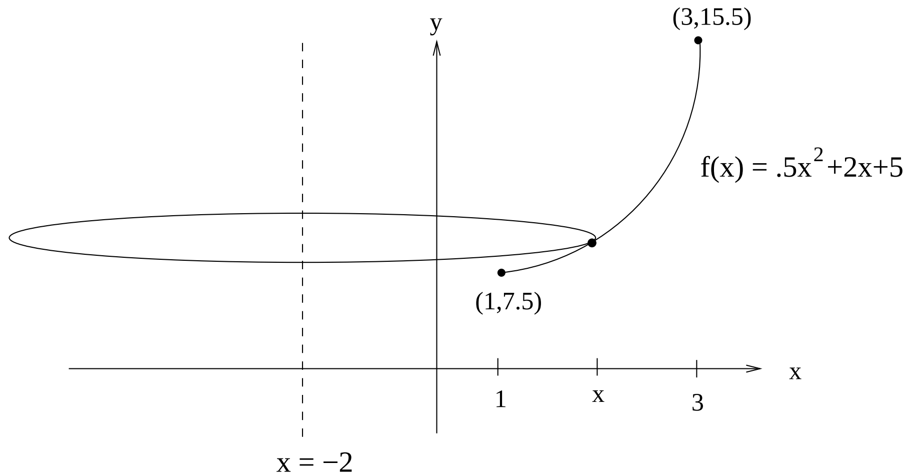
Problem	1	2	3	4	Total
Possible Score	30	45	75	50	200
Your Score					

SHOW ALL WORK. Any solution that is not accompanied by the appropriate work necessary for solving the problem will receive no credit. Do not use your calculator to evaluate any limits, derivatives, or integrals. If you need more space, you may use the back of the page.

TRIG IDENTITIES

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\sec(\theta) = \dfrac{1}{\cos(\theta)} \Rightarrow \cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)} \Rightarrow \sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

1. (30 pts) Let $\mathcal{C}$ be the curve $y = \frac{1}{2}x^2 + 2x + 5$ for $1 \le x \le 3$. Find the area of the surface obtained by rotating $\mathcal{C}$ around the line $x = -2$.



SOLUTION: $A = 2\pi \int_1^3 r(x)\sqrt{1+(y')^2}\,dx$

$$= 2\pi \int_1^3 (x+2)\sqrt{1+(x+2)^2}\,dx$$

Let $u = 1 + (x+2)^2 \Rightarrow du = 2(x+2)dx \Rightarrow dx = \dfrac{du}{2(x+2)}$, so we have

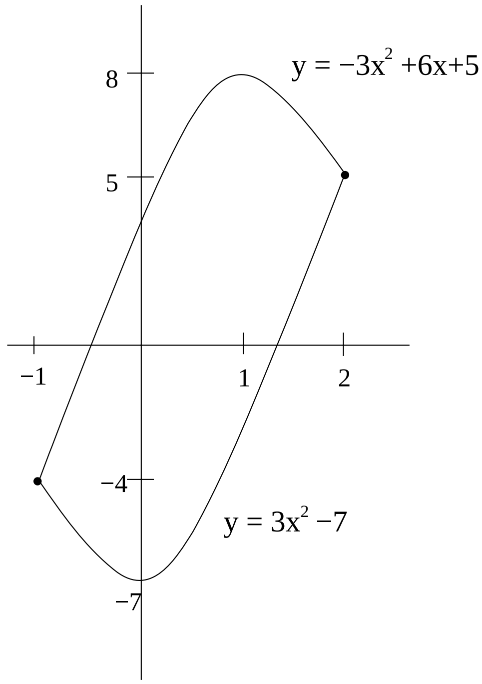
$$= 2\pi \int_{10}^{26} (x+2)\sqrt{u} \cdot \frac{du}{2(x+2)}$$

$$= \pi \int_{10}^{26} u^{1/2}\,du$$

$$= \pi \left.\frac{2}{3}u^{3/2}\right|_{10}^{26}$$

$$\boxed{= \frac{2\pi}{3}\left((26)^{3/2} - (10)^{3/2}\right) \approx 211.43}$$

2. Let $\mathcal{R}$ be the region bounded by the curves $y = 3x^2 - 7$ and $y = -3x^2 + 6x + 5$ (see the below figure).



(a) (15 pts) Find the area of $\mathcal{R}$.

SOLUTION: $A = \displaystyle\int_{-1}^{2} (-3x^2 + 6x + 5) - (3x^2 - 7)\, dx$

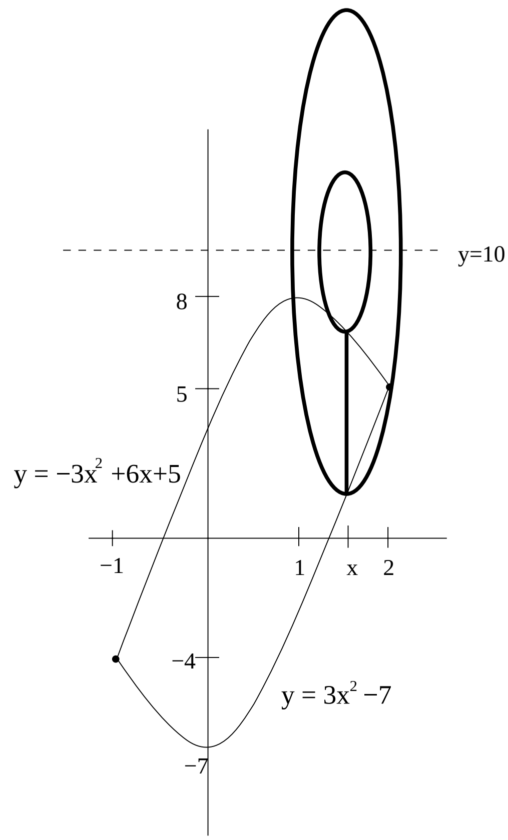
$$= \int_{-1}^{2} -6x^2 + 6x + 12\, dx$$

$$= \Big|_{-1}^{2} - 2x^3 + 3x^2 + 12x$$

$$= -16 + 12 + 24 - (2 + 3 - 12)$$

$$\boxed{= 27}$$

(b) (30 pts) Find the volume of the solid obtained by rotating $\mathcal{R}$ around the line $y = 10$.



SOLUTION: $V = \pi \int_{-1}^{2} (R(x))^2 - (r(x))^2 \, dx$

$$= \pi \int_{-1}^{2} \left(10 - (3x^2 - 7)\right)^2 - \left(10 - (-3x^2 + 6x + 5)\right)^2 \, dx$$

$$= \pi \int_{-1}^{2} (9x^4 - 102x^2 + 289) - (9x^4 - 36x^3 + 66x^2 - 60x + 25) \, dx$$

$$= \pi \int_{-1}^{2} 36x^3 - 168x^2 + 60x + 264 \, dx$$

$$= \pi \left. 9x^4 - 56x^3 + 30x^2 + 264x \right|_{-1}^{2}$$

$$= \pi \big[(144 - 448 + 120 + 528) - (9 + 56 + 30 - 264) \big]$$

$$\boxed{= 513\pi \approx 1611.64}$$

3. Evaluate the following limits.

(a) (25 pts) $\lim_{x\to 2^+} \frac{\sqrt{x^2-3x+2}}{\ln(5-x^2)}$

SOLUTION: $\lim_{x\to 2^+} \frac{\sqrt{x^2-3x+2}}{\ln(5-x^2)} = \frac{0}{0}$, so we use L'Hospital's Rule.

$$\lim_{x\to 2^+} \frac{\sqrt{x^2-3x+2}}{\ln(5-x^2)} \stackrel{LH}{=} \lim_{x\to 2^+} \frac{\frac{1}{2}(x^2-3x+2)^{-1/2}(2x-3)}{\frac{-2x}{5-x^2}}$$

$$= \lim_{x\to 2^+} \frac{2x-3}{2\sqrt{x^2-3x+2}} \cdot \frac{5-x^2}{-2x}$$

$$= \frac{1\cdot 1}{2\cdot 0^+ \cdot -4}$$

$$\boxed{= -\infty}$$

(b) (25 pts) $\lim_{x\to 0^+} \tan(x)\ln(x)$

SOLUTION: $\lim_{x\to 0^+} \tan(x)\ln(x) = 0\cdot -\infty$, so we need to use L'Hospital's Rule.

$$\lim_{x\to 0^+} \tan(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\cot(x)}$$

$$\stackrel{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc^2(x)}$$

$$= \lim_{x\to 0^+} -\frac{\sin^2(x)}{x}$$

$$\stackrel{LH}{=} \lim_{x\to 0^+} -\frac{2\sin(x)\cos(x)}{1}$$

$$\boxed{= 0}$$

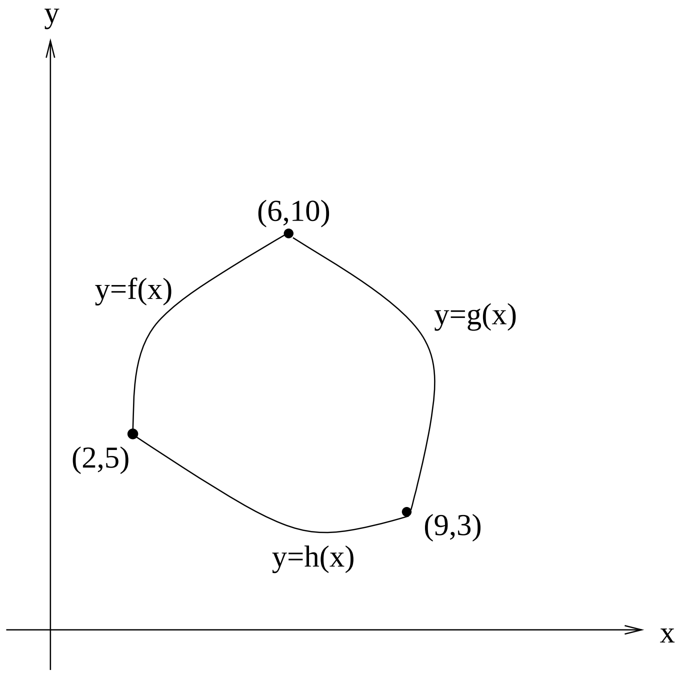
(c) (25 pts) $\lim_{x\to-\infty}\left(\frac{2x-5}{3x^2+1}\right)^{e^x}$

SOLUTION: $\lim_{x\to-\infty}\left(\frac{2x-5}{3x^2+1}\right)^{e^x}=0^0$, so we let $L=\lim_{x\to-\infty}\left(\frac{2x-5}{3x^2+1}\right)^{e^x}$.

$$\begin{aligned}
\text{Thus, } \ln(L) &= \lim_{x\to-\infty} \ln\left(\left(\frac{2x-5}{3x^2+1}\right)^{e^x}\right) \\
&= \lim_{x\to-\infty} e^x \ln\left(\frac{2x-5}{3x^2+1}\right) \\
&= \lim_{x\to-\infty} \frac{\ln\left(\frac{2x-5}{3x^2+1}\right)}{e^{-x}} \\
&\overset{LH}{=} \lim_{x\to-\infty} \frac{\frac{3x^2+1}{2x-5}\cdot\frac{(3x^2+1)2-(2x-5)6x}{(3x^2+1)^2}}{-e^{-x}} \\
&= \lim_{x\to-\infty} -e^x\left(\frac{-6x^2+30x+2}{(2x-5)(3x^2+1)}\right) \\
&= 0
\end{aligned}$$

Since $\ln(L)=0$, we have $\boxed{L=1}$

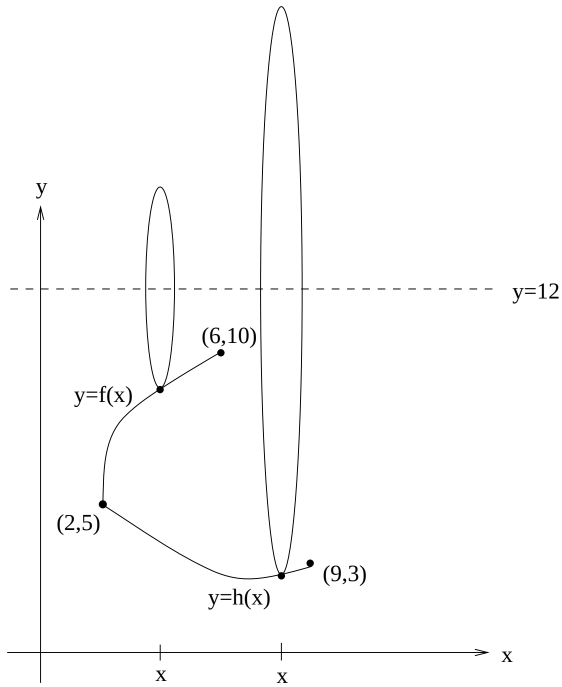
4. The following graph contains three curves: $y = h(x)$ for $2 \le x \le 9$ is the bottom curve, $y = f(x)$ for $2 \le x \le 6$ is the left curve, and $y = g(x)$ for $6 \le x \le 9$ is the right curve (see the figure below).



(a) (10 pts) Let $\mathcal{C}_1$ be the curve $y = f(x)$ for $2 \le x \le 6$ together with $y = g(x)$ for $6 \le x \le 9$ (so these two curves together make up $\mathcal{C}_1$). Set up but DO NOT EVALUATE the integral(s) for the arc length of $\mathcal{C}_1$.

SOLUTION: $$\boxed{L = \int_2^6 \sqrt{1 + (f'(x))^2}\, dx + \int_6^9 \sqrt{1 + (g'(x))^2}\, dx}$$

(b) (15 pts) Let $\mathcal{C}_2$ be the curve $y = f(x)$ for $2 \le x \le 6$ together with $y = h(x)$ for $2 \le x \le 9$ (so these two curves together make up $\mathcal{C}_2$). Set up but DO NOT EVALUATE the integral(s) for the area of the surface obtained by rotating $\mathcal{C}_1$ around the line $y = 12$.

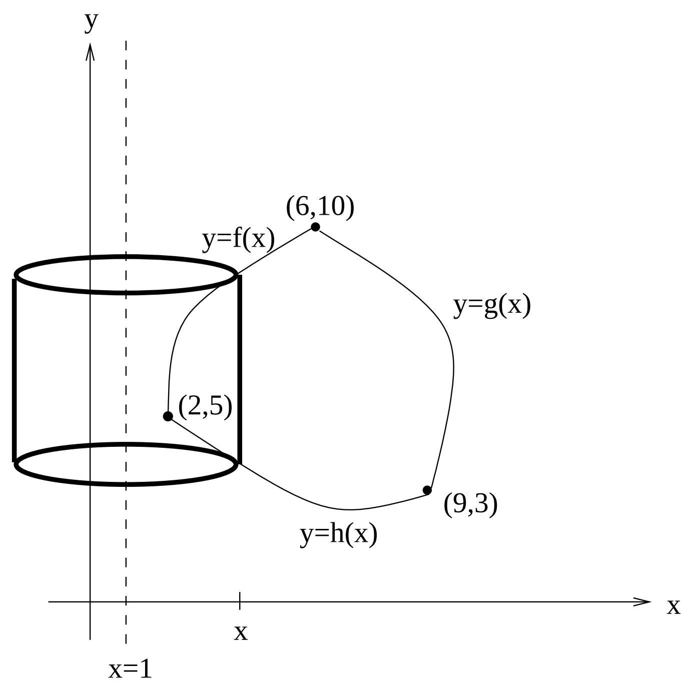


SOLUTION: $A = 2\pi \int_2^6 (12 - f(x))\sqrt{1 + (f'(x))^2}\, dx + 2\pi \int_2^9 (12 - h(x))\sqrt{1 + (h'(x))^2}\, dx$

(c) (10 pts) Let $\mathcal{R}$ be the region bounded by $y = f(x)$, $y = g(x)$, and $y = h(x)$. Set up but DO NOT EVALUATE the integral(s) for the area of $\mathcal{R}$.

SOLUTION: $A = \int_2^6 f(x) - h(x)\, dx + \int_6^9 g(x) - h(x)\, dx$

(d) (15 pts) Set up but DO NOT EVALUATE the integral(s) for the volume of the solid obtained by rotating $\mathcal{R}$ around the line $x = 1$.



SOLUTION: $V = 2\pi \int_2^6 (x-1)\big(f(x) - h(x)\big)\, dx + 2\pi \int_6^9 (x-1)\big(g(x) - h(x)\big)\, dx$