TEST #1

Math 142 **Name:** ______________________________

Problem	1	2	3	4	5	Total
Possible Score	40	40	40	40	40	200
Your Score						

SHOW ALL WORK. Any solution that is not accompanied by the appropriate work necessary for solving the problem will receive no credit. Do not use your calculator to evaluate any limits, derivatives, or integrals. If you need more space, you may use the back of the page.

TRIG IDENTITIES

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\sec(\theta) = \dfrac{1}{\cos(\theta)} \Rightarrow \cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)} \Rightarrow \sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

1. (40 pts) Evaluate $\displaystyle\int \frac{5x^2 - 16x + 13}{(3x-1)(x^2+4)}\,dx$

SOLUTION: We need to you partial fractions.

$$\frac{5x^2 - 16x + 13}{(3x-1)(x^2+4)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + (Bx+C)(3x-1)}{(3x-1)(x^2+4)}$$

Thus, $5x^2 - 16x + 13 = A(x^2+4) + (Bx+C)(3x-1)$.

If $x = \frac{1}{3}$, we get $\frac{74}{9} = \frac{37}{9}A \Rightarrow A = 2$.

If $x = 0$, we get $13 = 8 - C \Rightarrow C = -5$.

If $x = 1$, we get $2 = 10 + 2B - 10 \Rightarrow B = 1$.

Therefore, $\displaystyle\int \frac{5x^2 - 16x + 13}{(3x-1)(x^2+4)}\,dx = \int \frac{2}{3x-1} + \frac{x-5}{x^2+4}\,dx$

$$= 2\int \frac{1}{3x-1}\,dx + \int \frac{x}{x^2+4}\,dx - 5\int \frac{1}{x^2+4}\,dx$$

For the middle integral, we use u-substitution with $u = x^2 + 4 \Rightarrow dx = \frac{du}{2x}$, so we get

$$= 2\int \frac{1}{3x-1}\,dx + \int \frac{x}{u}\frac{du}{2x} - 5\int \frac{1}{x^2+4}\,dx$$

$$= 2\int \frac{1}{3x-1}\,dx + \frac{1}{2}\int \frac{1}{u}\,du - 5\int \frac{1}{x^2+4}\,dx$$

$$= \frac{2}{3}\ln|3x-1| + \frac{1}{2}\ln|u| - \frac{5}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$\boxed{= \frac{2}{3}\ln|3x-1| + \frac{1}{2}\ln(x^2+4) - \frac{5}{2}\arctan\left(\frac{x}{2}\right) + C}$$

2. (40 pts) Evaluate ONE of the following:

(a) $\int x^3 \sqrt{x+2}\, dx$

(b) $\int \tan^3(x) \sin^2(x) \sec^2(x)\, dx$

(c) $\int \frac{6}{25x^2 + 20x + 8}\, dx$ (**Hint:** complete the square)

SOLUTION FOR (a): Let $u = x + 2 \Rightarrow du = dx$, so $x = u - 2 \Rightarrow x^3 = (u-2)^3$.

$$\begin{aligned}
\int x^3 \sqrt{x+2}\, dx &= \int (u-2)^3 \sqrt{u}\, du \\
&= \int (u^3 - 6u^2 + 12u - 8)u^{1/2}\, du \\
&= \int u^{7/2} - 6u^{5/2} + 12u^{3/2} - 8u^{1/2}\, du \\
&= \frac{2}{9}u^{9/2} - \frac{12}{7}u^{7/2} + \frac{24}{5}u^{5/2} - \frac{16}{3}u^{3/2} + C \\
&\boxed{= \frac{2}{9}(x+2)^{9/2} - \frac{12}{7}(x+2)^{7/2} + \frac{24}{5}(x+2)^{5/2} - \frac{16}{3}(x+2)^{3/2} + C}
\end{aligned}$$

SOLUTION FOR (b): $$\int \tan^3(x)\sin^2(x)\sec^2(x)\,dx = \int \frac{\sin^3(x)}{\cos^3(x)} \cdot \sin^2(x) \cdot \frac{1}{\cos^2(x)}\,dx$$

$$= \int \frac{\sin^5(x)}{\cos^5(x)}\,dx$$

Let $u = \cos(x) \Rightarrow dx = -\dfrac{du}{\sin(x)}$

$$= -\int \frac{\sin^5(x)}{u^5} \cdot \frac{du}{\sin(x)}$$

$$= -\int \frac{\sin^4(x)}{u^5}\,du$$

$$= -\int \frac{(1-\cos^2(x))^2}{u^5}\,du$$

$$= -\int \frac{(1-u^2)^2}{u^5}\,du$$

$$= -\int \frac{1-2u^2+u^4}{u^5}\,du$$

$$= -\int u^{-5} - 2u^{-3} + u^{-1}\,du$$

$$= -\left(-\frac{1}{4}u^{-4} + u^{-2} + \ln|u|\right) + C$$

$$\boxed{= \frac{1}{4}\sec^4(x) - \sec^2(x) - \ln|\cos(x)| + C}$$

SOLUTION FOR c: We complete the square to get

$$25x^2 + 20x + 8 = (ax + b)^2 + c = a^2x^2 + 2abx + (b^2 + c).$$

Thus, $a^2 = 25 \Rightarrow a = 5$.

$2ab = 20 \Rightarrow 10b = 20 \Rightarrow b = 2$.

$b^2 + c = 8 \Rightarrow 4 + c = 8 \Rightarrow c = 4$.

Therefore, $\displaystyle\int \frac{6}{25x^2 + 20x + 8}\,dx = \int \frac{6}{(5x + 2)^2 + 4}\,dx$

Let $u = 5x + 2 \Rightarrow dx = \dfrac{du}{5}$.

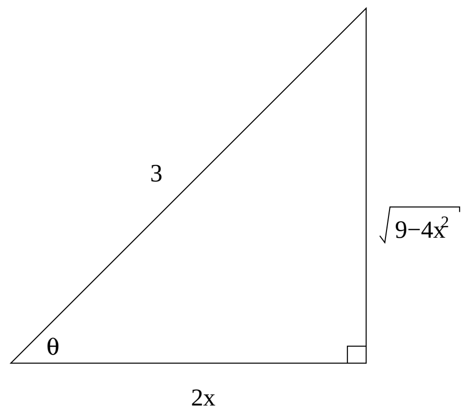
$$= \int \frac{6}{u^2 + 4} \cdot \frac{du}{5}$$

$$= \frac{6}{5} \int \frac{1}{u^2 + 4}\,du$$

$$= \frac{3}{5} \arctan\left(\frac{u}{2}\right) + C$$

$$\boxed{= \frac{3}{5} \arctan\left(\frac{5x + 2}{2}\right) + C}$$

3. (40 pts) Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-4x^2}}\,dx$



SOLUTION: From the above right triangle, we see that

$$x = \frac{3}{2}\cos(\theta) \Rightarrow dx = -\frac{3}{2}\sin(\theta)d\theta$$

$\sqrt{4x^2-9} = 3\sin(\theta)$, so we have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-4x^2}}\,dx &= -\int \frac{(\frac{3}{2}\cos(\theta))^2}{3\sin(\theta)} \cdot \frac{3}{2}\sin(\theta)d\theta \\
&= -\frac{9}{8}\int \cos^2(\theta)\,d\theta \\
&= -\frac{9}{16}\int 1+\cos(2\theta)\,d\theta \\
&= -\frac{9}{16}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&= -\frac{9}{16}\big(\theta + \sin(\theta)\cos(\theta)\big) + C \\
&\boxed{= -\frac{9}{16}\arccos\left(\frac{2x}{3}\right) - \frac{2x\sqrt{9-4x^2}}{16} + C}
\end{aligned}$$

4. (40 pts) Evaluate $\int e^{3x}(x^2 - 4x + 5)\, dx$

SOLUTION: We use integration by parts with $u = x^2 - 4x + 5$, $dv = e^{3x}dx$, $du = (2x - 4)dx$, and $v = \frac{1}{3}e^{3x}$.

Therefore, $\int e^{3x}(x^2 - 4x + 5)\, dx = \frac{1}{3}e^{3x}(x^2 - 4x + 5) - \frac{1}{3}\int e^{3x}(2x - 4)\, dx$

For this integral, we use integration by parts with $u = 2x - 4$, $dv = e^{3x}dx$, $du = 2dx$, and $v = \frac{1}{3}e^{3x}$.

$$= \frac{1}{3}e^{3x}(x^2 - 4x + 5) - \frac{1}{3}\left(\frac{1}{3}e^{3x}(2x - 4) - \frac{2}{3}\int e^{3x}\, dx\right)$$

$$\boxed{= \frac{1}{3}e^{3x}(x^2 - 4x + 5) - \frac{1}{9}e^{3x}(2x - 4) + \frac{2}{27}e^{3x} + C}$$

5. (40 pts) Evaluate ONE of the following:

(a) $\displaystyle\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int \frac{dx}{\sqrt{4x^2-4x+5}}$ (**Hint:** complete the square)

(c) $\displaystyle\int \frac{3x-4}{2+\sqrt{x}}\,dx$

SOLUTION FOR a: Let $u=\sqrt{1-x^2} \Rightarrow u^2 = 1-x^2 \Rightarrow x^2 = 1-u^2 \Rightarrow xdx = -udu.$

$$\text{Thus, } \int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx = -\int \frac{u}{u^2+u}\,du$$

$$= -\int \frac{u}{u(u+1)}\,du$$

$$= -\int \frac{1}{u+1}\,du$$

$$= -\ln|u+1| + C$$

$$\boxed{= -\ln(\sqrt{1-x^2}+1) + C}$$

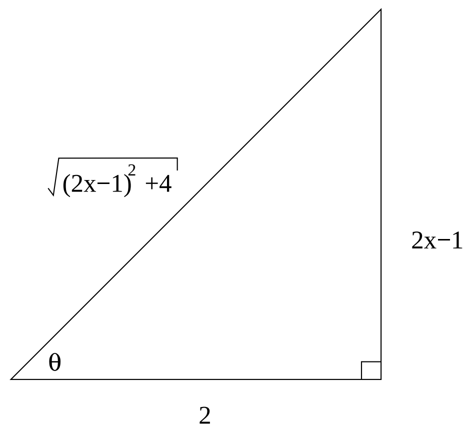
SOLUTION FOR b: We complete the square to get

$$4x^2 - 4x + 5 = (ax+b)^2 + c = a^2x^2 + 2abx + (b^2 + c), \text{ so } a^2 = 4 \Rightarrow a = 2.$$

$$2ab = -4 \Rightarrow 4b = -4 \Rightarrow b = -1.$$

$b^2 + c = 5 \Rightarrow 1 + c = 5 \Rightarrow c = 4$, so we have

$$\int \frac{dx}{\sqrt{4x^2 - 4x + 5}} = \int \frac{dx}{\sqrt{(2x-1)^2 + 4}}$$



We can see from the above right triangle that

$$\frac{2x-1}{2} = \tan(\theta) \Rightarrow x = \tan(\theta) + \frac{1}{2} \Rightarrow dx = \sec^2(\theta) d\theta$$

$\sqrt{(2x-1)^2 + 4} = 2\sec(\theta)$, so we have

$$\int \frac{dx}{\sqrt{(2x-1)^2 + 4}} = \int \frac{\sec^2(\theta)}{2\sec(\theta)}\, d\theta$$

$$= \frac{1}{2} \int \sec(\theta)\, d\theta$$

$$= \frac{1}{2} \ln|\sec(\theta) + \tan(\theta)| + C$$

$$\boxed{= \frac{1}{2} \ln\left|\frac{\sqrt{4x^2 - 4x + 5} + 2x - 1}{2}\right| + C}$$

SOLUTION FOR c: Let $u = 2 + \sqrt{x} \Rightarrow x = (u-2)^2 \Rightarrow dx = 2(u-2)du$.

$$\begin{aligned}
\int \frac{3x-4}{2+\sqrt{x}}\,dx &= \int \frac{3(u-2)^2-4}{u}\cdot 2(u-2)\,du \\
&= 2\int \frac{3u^2-12u+8}{u}\cdot (u-2)\,du \\
&= 2\int \frac{3u^3-18u^2+32u-16}{u}\,du \\
&= 2\int 3u^2-18u+32-\frac{16}{u}\,du \\
&= 2\left(u^3-9u^2+32u-16\ln|u|\right)+C \\
&\boxed{= 2(2+\sqrt{x})^3-18(2+\sqrt{x})^2+64(2+\sqrt{x})-32\ln(2+\sqrt{x})+C}
\end{aligned}$$