A Modular Triple Characterization of Lifting Signatures, Weak Orientations, Orientations, and Ternary Signatures of Matroids

Lori Koban

Department of Mathematics and Computer Science

Western Carolina University

Email: lkoban@email.wcu.edu

Phone: (828)227-2484

Fax: (828) 227-7240

Address: WCU, Dept. of Math & C.S., Cullowhee, NC 28723

Abstract

For a matroid M, Dowling and Kelly show how to use a special class of its circuits, which we call *balanced circuits*, to construct a lift of M. In the case where M comes from a graph, Zaslavsky defines balanced circuits in terms of *gains*, which are group elements that label the edges of the graph. I will use gains to define the balanced circuits of an arbitrary matroid and show that the lift construction can be done precisely when M is ternary. Instrumental to the proof is a theorem that uses modular triples of circuits to characterize four types of circuit signatures, three of which are known (weak orientations, orientations, and ternary signatures) and one of which is new (lifting signatures).

1. Introduction

This paper contains two main theorems that were inspired by the theories of matroids, oriented matroids, and gain graphs. The first, Theorem 3.1, uses the notion of *modular triples of circuits* (a generalization of theta graphs) to characterize four types of circuit signatures of matroids. The second main result is the combination of Theorems 4.4 and 4.5. They say that it is possible to construct elementary lifts of a ternary (Theorem 4.4) or binary (Theorem 4.5) matroid by labeling its elements with members of certain groups. Our two main results are related insofar as Theorem 3.1 is instrumental to the proof of Theorem 4.4 and was, in part, conceived for this purpose.

The lift matroids in Theorems 4.4 and 4.5 arise as a generalization of a lift construction due to Zaslavsky [17, Section 3], which is an application of a lift construction due to Dowling and Kelly [6, Section 6]. These constructions require a special type of class of matroid circuits, called a *linear* class. Given a matroid M and a linear class $\mathcal{B}$ of its circuits, Dowling and Kelly described how $\mathcal{B}$ determines an elementary lift of M. Zaslavsky applied their construction in the case where M is a graphic matroid. He labeled the edges of the associated graph with group elements, called *gains*. The gain of an edge is arbitrarily associated with a fixed orientation of that edge. If the edge is traversed in the opposite direction, then the gain is the inverse of the group element. Zaslavsky defined $\mathcal{B}$, the set of *balanced circles*, to consist of those edge sets of circles (simple closed paths) for which 1 is the product of the gains of the edges as they are traversed around the circle. In this way, gains are used to construct lifts of graphic matroids.

We want to use gains to construct lifts of more matroids than graphic matroids. This involves labeling the elements of a matroid with gains and defining the notion of a balanced circuit. In order to define a balanced circuit, we need a way to distinguish between the gain that is associated with a matroid element and its inverse. In a graph, this feat is accomplished by direction, as explained above. Our substitute for direction is circuit signatures of matroids.

This is appropriate because Zaslavsky's definition of a balanced circle has an interpretation in terms of circuit signatures. Arbitrarily orient both the edges of a graph and its circles. An element of a signed circuit is positive if the orientations of the edge and the circle agree and is negative if they disagree. This defines a circuit signature of the graphic matroid. (This is a special type of circuit signature, called an orientation.) A circle is balanced (has unit gain) precisely when the product of the gains of the positive elements and the inverses of the gains of the negative elements equals 1. This method of defining balanced circles generalizes easily to a definition of balanced circuits for arbitrary matroids.

Given a matroid whose edges are labeled by gains and given a signature of its circuits, let $\mathcal{B}$ be the set of *balanced circuits*, those circuits with unit gain. Dowling and Kelly's lift construction applies only when $\mathcal{B}$ is linear, which inspires the following definition: a *lifting signature* is a circuit signature that forces $\mathcal{B}$ to be linear. Thus gains can be used to lift a matroid if and only if it has a lifting signature. Now we ask, which matroids have lifting signatures? The answer, according to Theorems 4.4 and 4.5, is the ternary and binary matroids.

To prove that ternary matroids can be lifted using gains, we apply Theorem 3.1 and find that ternary signatures [12] (which are a particular kind of circuit signature for ternary matroids) and lifting signatures coincide. I originally proved Theorem 3.1 for lifting signatures. Then I found that modular triples also characterize weak orientations [1], orientations [2], and ternary signatures [12]. Our characterization of these circuit signatures in terms of modular triples is new. Many other characterizations already exist, such as by forbidden minors, circuit elimination, and orthogonality. The following chart shows where these characterizations appear in this . The proof of Theorem 3.1 employs some aspects of the circuit elimination and forbidden minor characterizations.

	Orientations	chapter Weak Orientations	Ternary/Lifting Signatures
Forbidden minors	[9, Theorem 2.1]	[8, Theorem 1, p. 173]	[12, Theorem 3.1]
Circuit elimination	[2, Section 2]	[1, Theorem 6.1]	[12, Theorem 3.1]
Orthogonality	[2, Theorem 2.2]	[1, Theorem 1.10]	[12, Theorem 3.1]
Modular triples	Thm. 3.1(2)	Thm. 3.1(1)	Thm. 3.1(3) and (4)

Though I created Theorem 3.1 to help understand lifting signatures, it has other applications. For example, it allows us to provide quick, easy proofs of several known facts about circuit signatures (see Corollary 5.1).

The rest of this paper consists of four sections. Section 2 contains background information. In Section 3, we use modular triples to characterize weak orientations, orientations, and ternary signatures. The characterization of lifting signatures by modular triples appears in Section 4. In that section, we also answer the question: Which matroids can be lifted using gains? In Section 5, we give some applications of our main theorems.

2. Background

2.1. **Using Linear Classes of Circuits to Construct Matroid Lifts.** Let M be a matroid with ground set E. In [6], Dowling and Kelly explain how to use a linear class of circuits to construct an elementary lift of M. In this section, we discuss the construction, but we use different terminology. Most of our notation and terminology is from [11].

2.1.1. *Modular Triples.* We say that (C_1, C_2, C_3) is a *modular triple of circuits* of M if the three circuits are distinct and, for distinct i, j, and k, $C_k \subseteq C_i \cup C_j$ and (C_i, C_j) is a modular pair. $\big($Recall that (C_i, C_j) is a *modular pair of circuits* if $r(C_i \cup C_j) = |C_i \cup C_j| - 2.\big)$ A generalization of this concept appears in [1, Section 5], and our definition is equivalent to the one in [4, Section 7.1]. If M is a graphic matroid, then a modular triple of circuits consists of the three circles of a theta graph. Thus a modular triple of circuits is a matroid generalization of a theta graph.

We say that (H_1, H_2, H_3) is a *modular triple of copoints* of M if the three copoints are distinct and intersect in a coline. (Copoints and colines are flats whose ranks are less than the rank of M by 1 and 2, respectively.) In other words, for distinct i, j, and k, $H_i \cap H_j \subseteq H_k$ and (H_i, H_j) is a modular pair. (Recall that (F_i, F_j) is a *modular pair of flats* if $r(F_i) + r(F_j) = r(F_i \cup F_j) + r(F_i \cap F_j)$.) This definition appears in [4, Section 7.1].

It is not hard to show that (C_1, C_2, C_3) is a modular triple of circuits of M if and only if $(E\backslash C_1, E\backslash C_2, E\backslash C_3)$ is a modular triple of copoints of M^*. We usually write H_i^* instead of $E\backslash C_i$. If L^* is the dual coline (i.e., in M^*) at which H_1^*, H_2^*, and H_3^* meet, then $L^* = E\backslash(C_1 \cup C_2 \cup C_3)$. Figure 2.1 is helpful when thinking about modular triples. An easy, yet important, observation is that I_{13}, I_{12}, and I_{23} are nonempty (because the three circuits are distinct).

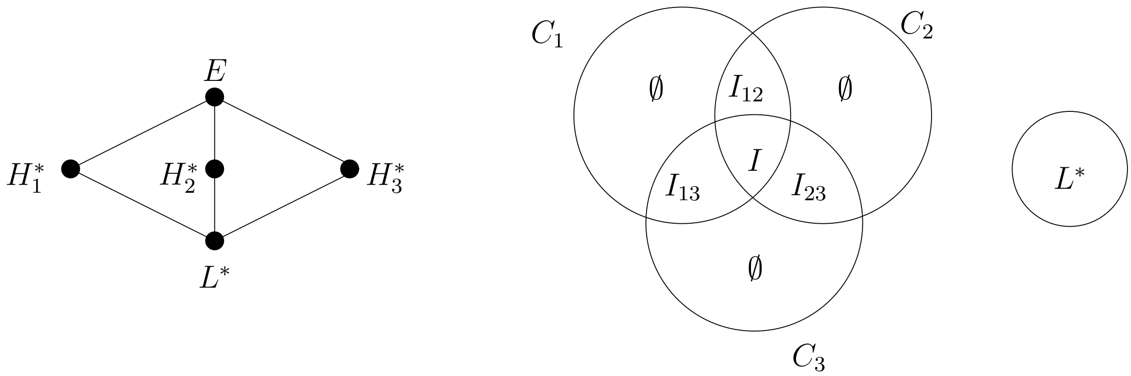


FIGURE 2.1. On the left is a portion of the lattice of flats of M^* in which (H_1^*, H_2^*, H_3^*) is a modular triple of copoints. On the right we see (C_1, C_2, C_3), the corresponding modular triple of circuits of M.

Throughout this paper, whenever (C_1, C_2, C_3) is a modular triple of circuits, the sets I, I_{13}, I_{12}, and I_{23} are defined as in Figure 2.1.

At this point, we include two useful lemmas about modular triples of circuits.

Lemma 2.1. *A matroid M is binary if and only if, for each modular triple (C_1, C_2, C_3) of circuits, $C_1 \cap C_2 \cap C_3 = \emptyset$.*

Proof. Let (C_1, C_2, C_3) be a modular triple of circuits of M, and suppose $w \in C_1 \cap C_2 \cap C_3$. Thus $w \notin H_1^* \cup H_2^* \cup H_3^*$. Also, let L^* be the coline at which these copoints meet, so $w \notin L^*$. Since the copoints in the interval $[L^*, E]$ partition the elements of $E \backslash L^*$, there exists another copoint in this interval, say H_4^*, that contains w. But then the lattices of flats of $M^*|(H_1^* \cup H_2^* \cup H_3^* \cup H_4^*)/L^*$ and $U_{2,4}$ are isomorphic, which means that M^* has a $U_{2,4}$ minor. Equivalently, M has a $U_{2,4}$ minor.

Now assume that M is not binary. Thus M^* has a $U_{2,4}$ minor. By the Scum Theorem, M^* has a coline L^* such that $[L^*, E]$ contains four distinct copoints, namely H_1^*, H_2^*, H_3^*, and H_4^*. Since (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of M^*, (C_1, C_2, C_3) is a modular triple of circuits of M. Let $w \in (H_4^* \backslash L^*)$, so $w \notin H_1^* \cup H_2^* \cup H_3^*$. Thus $w \in C_1 \cap C_2 \cap C_3$. □

Lemma 2.2. *Let (C_1, C_2) be a modular pair of circuits of a matroid M, and let $e \in C_1 \cap C_2$. Then there exists a unique circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \backslash \{e\}$. Moreover, (C_1, C_2, C_3) is a modular triple of circuits of M.*

Proof. We know that (H_1^*, H_2^*) is a modular pair of copoints of M^*. Assume they meet at the coline L^*. Since $e \in C_1 \cap C_2$, it follows that $e \notin H_1^* \cup H_2^*$. Since the copoints in the interval $[L^*, E]$ partition the elements of $E \backslash L^*$, there exists a third copoint in this interval, say H_3^*, that contains e. Thus $e \notin C_3$. Since (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of M^*, (C_1, C_2, C_3) is a modular triple of circuits of M. Therefore, $C_3 \subseteq (C_1 \cup C_2)$.

Suppose $C_3' \subseteq (C_1 \cup C_2) \backslash \{e\}$. Then it corresponds to a copoint in the interval $[L^*, E]$ that contains e. But there is exactly one such copoint, namely H_3^*. Thus $C_3' = C_3$, which proves uniqueness. □

2.1.2. *Linear Subclasses.* Let $\mathcal{B}$ be a subclass of circuits of M. If, for each modular triple of circuits, either 0, 1, or 3 of these circuits are in $\mathcal{B}$, we say that $\mathcal{B}$ is a *linear subclass of circuits* of M. An equivalent definition appears in [17, Section 3].

Similarly, let $\mathcal{H}$ be a subclass of copoints of M. If, for each modular triple of copoints, either 0, 1, or 3 of these copoints are in $\mathcal{H}$, we say that $\mathcal{H}$ is a *linear subclass of copoints* of M. Crapo gave this definition in [5, Section 6].

It is easy to show that $\mathcal{B}$ is a linear subclass of circuits of M if and only if $\{E\backslash C : C \in \mathcal{B}\}$ is a linear subclass of copoints of M^*.

My definitions of linear subclasses of circuits and copoints are different from those in the literature because they are stated in terms of modular triples.

2.1.3. *Single-Element Extensions and Modular Cuts.* Let M and N be matroids. We say that N is a *single-element extension of* M if $M = N\backslash\{e\}$.

There is a one-to-one correspondence between single-element extensions and modular cuts of M [11, Section 7.2]. A set $\mathcal{M}$ of flats of M is called a *modular cut* if it satisfies the following rules:

(1) if $F \in \mathcal{M}$ and F' is a flat containing F, then $F' \in \mathcal{M}$, and
(2) if F_1, $F_2 \in \mathcal{M}$ and (F_1, F_2) is a modular pair of flats, then $F_1 \cap F_2 \in \mathcal{M}$.

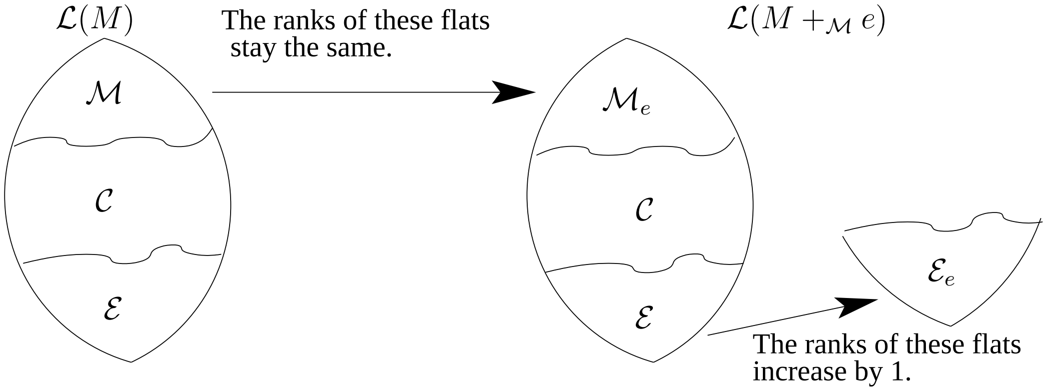
If $M = N\backslash\{e\}$ and $\mathcal{M}$ is the modular cut corresponding to the single-element extension N, we write N as $M +_{\mathcal{M}} e$. Denote the set of flats of M by $\mathcal{F}(M)$. Figure 2.2 illustrates how $\mathcal{F}(M +_{\mathcal{M}} e)$ is related to $\mathcal{F}(M)$. In this figure,

$$\mathcal{C} = \{F \in \mathcal{F}(M) : F \notin \mathcal{M} \text{ and } F \text{ is covered by an element of } \mathcal{M}\},$$

and

$$\mathcal{E} = \mathcal{F}(M)\backslash(\mathcal{M} \cup \mathcal{C}).$$

Figure 2.2. This figure illustrates the close connection between $\mathcal{L}(M)$, the lattice of flats of M, and $\mathcal{L}(M +_{\mathcal{M}} e)$, the lattice of flats of $M +_{\mathcal{M}} e$. Here, $\mathcal{M}_e = \{F \cup \{e\} : F \in \mathcal{M}\}$; and $\mathcal{E}_e$ is defined similarly.



Now it is easy to see that

$$r(M +_{\mathcal{M}} e) = \begin{cases} r(M) + 1 & \text{if } \mathcal{M} = \emptyset, \\ r(M) & \text{otherwise.} \end{cases}$$

Also, $\mathcal{M} = \emptyset$ if and only if e is a coloop in $M +_{\mathcal{M}} e$, and $\mathcal{M} = \mathcal{F}(M)$ if and only if e is a loop in $M +_{\mathcal{M}} e$.

We say that a modular cut $\mathcal{M}$ of M is *nontrivial* if $\mathcal{M} \neq \emptyset$ and $\mathcal{M} \neq \mathcal{F}(M)$.

2.1.4. *Elementary Lifts.* A matroid L is an *elementary lift* of M if there exists a matroid N and an element e of N that is not a loop or coloop such that $N/\{e\} = M$ and $N\backslash\{e\} = L$. Since N is a single-element extension of L, $N = L +_{\mathcal{M}} e$. So if L is an elementary lift of M, then $M = (L +_{\mathcal{M}} e)/\{e\}$ for some nontrivial modular cut $\mathcal{M}$ of L. It is easy to see that the rank of M is one less than that of L and that their ground sets are the same.

We are ready to show how a linear subclass of circuits of M enables the construction of a unique elementary lift of M. Let $\mathcal{B}$ be such a subclass. Define $\mathcal{B}^* = \{E\backslash C : C \in \mathcal{B}\}$, so $\mathcal{B}^*$ is a linear subclass of copoints of M^*. Then

$$\mathcal{M}_0 = \{F \in \mathcal{F}(M^*) : \text{ every copoint containing } F \text{ is in } \mathcal{B}^*\} \tag{2.1}$$

is a modular cut of M^* $\big($see [5, Section 6]$\big)$. This modular cut is nontrivial as long as $\mathcal{B}$ does not contain all circuits of M. (It is impossible for $\mathcal{M}_0$ to be empty because $\mathcal{M}_0$ always contains E.) Assume that there is a circuit of M that is not in $\mathcal{B}$. Then M^* is an elementary lift of $\big(M^* +_{\mathcal{M}_0} e\big)/\{e\}$. Equivalently, $\big((M^* +_{\mathcal{M}_0} e)/\{e\}\big)^*$ is an elementary lift of M. We denote this elementary lift by $L(M, \mathcal{B})$. Different choices of $\mathcal{B}$ yield different elementary lifts of M.

2.2. **Using Gains to Lift Graphic Matroids.** A *gain graph* $\Phi = (\Gamma, \phi)$ consists of a graph Γ and a *gain mapping* ϕ from the edges of Γ into a group $\mathfrak{G}$, the *gain group.* We require that $\phi(e^{-1}) = \phi(e)^{-1}$, where e^{-1} means e with its orientation reversed. Thus $\phi(e)$ depends on the orientation of e, but neither orientation is preferred. A reference for gain graphs is [16, Section 5].

Associated with Φ is a class $\mathcal{B}(\Phi)$ of *balanced circles.* Let B be a circle of Γ. To decide whether or not B is balanced, choose an edge e_1 of B and a direction (clockwise or counterclockwise) to traverse B. Let e_1, e_2, $\ldots$, e_k be the edges of B in the order in which they are traversed, and let them be oriented in this direction. The *gain* of B is $\phi(B) = \phi(e_1)\phi(e_2)\cdots\phi(e_k)$. Then $B \in \mathcal{B}(\Phi)$ if $\phi(B) = 1$. Whether or not $B \in \mathcal{B}(\Phi)$ is independent of the choices of e_1 and the direction in which B is traversed.

An edge set of Γ is *balanced* if every circle in it is balanced, and is *contrabalanced* if it has no balanced circles.

Let Φ be the gain graph in Figure 2.3. In this example,

$$\mathcal{B}(\Phi) = \big\{\{a, e, d\}, \{d, f, c\}, \{a, e, f, c\}\big\}.$$

The balanced circles are the three circles of a theta graph. This is no coincidence; the set of balanced circles of a gain graph is a linear subclass of circuits of the graphic matroid $G(\Gamma)$ [16, Proposition 5.1]. It follows from Section 2.1 that $L\big(G(\Gamma), \mathcal{B}(\Phi)\big)$ is an elementary lift of

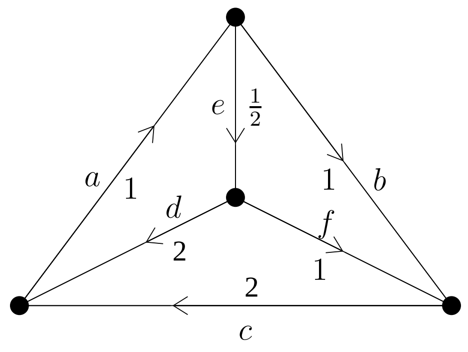


FIGURE 2.3. This is a gain graph with gain mapping $\phi : E(K_4) \to \mathbb{R}^*$.

$G(\Gamma)$ (unless all circles are balanced). In [17, Section 3], Zaslavsky refers to this lift as the *lift of* $G(\Gamma)$ *along* $\mathcal{B}(\Phi)$. We also denote it by $L(\Phi)$.

An easier description of $L(\Phi)$ is given in terms of its circuits, which are called *lift circuits* [16, Section 2]. A lift circuit is a balanced circle, a contrabalanced theta graph, a contrabalanced tight bracelet, or a contrabalanced loose bracelet. A *tight bracelet* is the union of two circles that intersect at precisely one vertex, and a *loose bracelet* is the union of two vertex-disjoint circles.

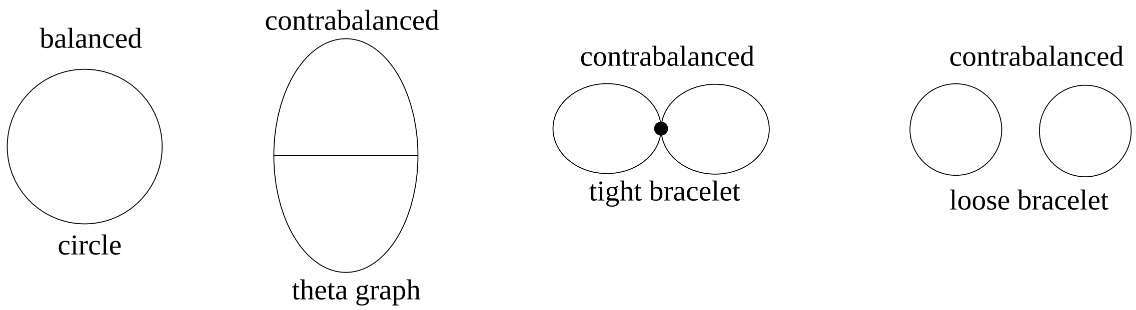


FIGURE 2.4. These are the four types of lift circuits.

If Φ is the gain graph in Figure 2.3, then the (lift) circuits of $L(\Phi)$ are

$$\{a, e, d\}, \{d, f, c\}, \text{ and } \{a, e, f, c\}.$$

$\big($In this example, the balanced circles of Φ happen to coincide with the circuits of $L(\Phi)$.$\big)$ The description of $L(\Phi)$ in 2.1 yields the same lift matroid, but the construction is more involved.

2.3. **Circuit Signatures.** A *signed set* is a set X together with an ordered bipartition of X into subsets X^+ and X^-. We denote the signed set by the ordered pair (X^+, X^-). The signed set has *underlying set* (or *support*) X. We denote both the signed set and its underlying set by X. Context indicates whether we are dealing with sets or signed sets.

If a signed set X has empty support, we say that $X = \emptyset$.

Sometimes, instead of denoting a signed set by $\big(\{a_1, \dots, a_p\}, \{b_1, \dots, b_n\}\big)$, we write it as $a_1 \cdots a_p \overline{b_1} \cdots \overline{b_n}$.

If X is a signed subset of E, then for each $e \in E$, define

$$X(e) = \begin{cases} +1 & \text{if } e \in X^+, \\ -1 & \text{if } e \in X^-, \\ 0 & \text{if } e \notin X^+ \cup X^-. \end{cases}$$

For every signed set X, we define the *negative* of X, denoted by $-X$, to be the signed set (X^-, X^+). We write $X = \pm Y$ if $X = Y$ or $X = -Y$. Also, $X \backslash T$ is the signed set $(X^+ \backslash T, X^- \backslash T)$.

Let $\mathcal{C}$ be a collection of signed sets. We define $\mathrm{Min}(\mathcal{C})$ to be the collection of elements of $\mathcal{C}$ with setwise minimal support.

Let M be a matroid on E, and let $\mathcal{C}$ be a collection of signed subsets of E. We say that $\mathcal{C}$ is a *circuit signature* of M if:

(1) every signed set in $\mathcal{C}$ has a circuit of M as underlying set, and

(2) for every circuit C of M, there are precisely two members of $\mathcal{C}$ with underlying set C, and these two signed sets are negatives of each other.

Frequently, when we write a circuit signature of M, we only specify one of the two signed sets with underlying set C for each circuit C of M.

Let $\mathcal{C}$ be a circuit signature of M. Define $-\mathcal{C} = \{-X \mid X \in \mathcal{C}\}$. Additional circuit signatures of M can be created by the process of reorientation. Given a signed set X in $\mathcal{C}$ and a set $A \subseteq E$, the *reorientation of* X *on* A, denoted by ${}_{\overline{A}}X$, is the signed set derived from X by reversing the signs of the elements of A. Technically,

$$ {}_{\overline{A}}X^{+} = \big(X^{+} \backslash A\big) \cup \big(X^{-} \cap A\big) $$

and

$$ {}_{\overline{A}}X^{-} = \big(X^{-} \backslash A\big) \cup \big(X^{+} \cap A\big). $$

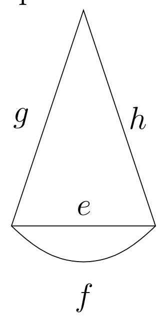
The *reorientation of* $\mathcal{C}$ *on* A is the circuit signature ${}_{\overline{A}}\mathcal{C} = \{{}_{\overline{A}}X : X \in \mathcal{C}\}$.

Let $e \in E$. Define $\mathcal{C}\backslash e = \{X \in \mathcal{C} : e \notin X\}$, the *deletion of* $\mathcal{C}$ *by* e. Also define the *contraction of* $\mathcal{C}$ *by* e, denoted by $\mathcal{C}/e$, to be the set $\operatorname{Min}\big\{X\backslash\{e\} : X \in \mathcal{C} \text{ and } X\backslash\{e\} \neq \emptyset\big\}$. Every collection obtained from $\mathcal{C}$ by a succession of deletions and contractions is called a *minor* of $\mathcal{C}$.

It is easy to see that $\mathcal{C}\backslash e$ and $\mathcal{C}/e$ are circuit signatures of $M\backslash e$ and M/e, respectively. Hence any minor of $\mathcal{C}$ is a circuit signature of the associated matroid minor. However, the order in which elements are deleted and contracted from $\mathcal{C}$ may affect the resulting minor. Consider $\mathcal{C} = \big\{eg\overline{h}, fgh, ef\big\}$, a circuit signature of the matroid of the graph in Figure 2.5. Then $\mathcal{C}/e/f = \big\{g\overline{h}\big\}$, but $\mathcal{C}/f/e = \{gh\}$.

Suppose that $\mathcal{C}$ is a circuit signature of M, and let C_1, C_2, and C_3 be elements of $\mathcal{C}$. We say that (C_1, C_2, C_3) is a *modular triple of signed circuits* if, as underlying sets, (C_1, C_2, C_3)

FIGURE 2.5. The matroid of this graph has a circuit signature that is not minorable.



is a modular triple of circuits of M. Make a similar definition for a *modular pair of signed circuits*.

2.4. **Orientations of Matroids.** Bland and Las Vergnas introduced oriented matroids in [2]. Just as matroids can be defined in many cryptomorphic ways, oriented matroids can be defined in terms of circuits, cocircuits, vectors, covectors, and more. We are most interested in the signed circuits of an oriented matroid, which we now define. A collection $\mathcal{C}$ of signed subsets of a set E is the set of *signed circuits* of an oriented matroid on E if it satisfies the following axioms:

(1) $\emptyset \notin \mathcal{C}$,
(2) (*Symmetry*) $\mathcal{C} = -\mathcal{C}$,
(3) (*Incomparability*) for all X, $Y \in \mathcal{C}$, if the support of X is contained in the support of Y, then $X = \pm Y$, and
(4) (*Elimination*) for all X, $Y \in \mathcal{C}$ such that $X \neq -Y$, and all $e \in X^+ \cap Y^-$, there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq \big(X^+ \cup Y^+\big)\backslash\{e\}$ and $Z^- \subseteq \big(X^- \cup Y^-\big)\backslash\{e\}$.

If M is a matroid on E with circuit signature $\mathcal{C}$, it is clear that $\mathcal{C}$ determines (is the set of signed circuits of) an oriented matroid if $\mathcal{C}$ also satisfies (4). In this case, we say that $\mathcal{C}$ is an *orientation* of M and that M is *orientable.* Las Vergnas showed that (4) only needs to be verified for modular pairs of signed circuits $\big($see [10, Theorem 2.1]$\big)$.

2.5. **Weak Orientations of Matroids.** Weakly oriented matroids were introduced by Bland and Jensen in [1]. Matroids that are weakly orientable are an intermediate class between all matroids and orientable matroids. Bland and Jensen first defined them in terms of the Minty Coloring Property. This property makes their definition of weakly oriented matroids seem natural (see [1, Theorems 1.4 and 1.5]).

Just as oriented matroids are matroids together with a special type of circuit signature, called an orientation, weakly oriented matroids are matroids together with a special type of circuit signature, called a weak orientation. Bland and Jensen give a characterization of weak orientations by circuit elimination [1, Theorem 6.1]. The following theorem is a restatement of their result.

Theorem 2.3. *Let $\mathcal{C}$ be a circuit signature of a matroid M. Then $\mathcal{C}$ is a weak orientation of M if and only if for every X_1, $X_2 \in \mathcal{C}$ with $e \in X_1^+ \cap X_2^-$ and $X_1 \neq -X_2$,*

(i) *if $f \in \big(X_1^+ \backslash X_2^-\big) \cup \big(X_1^- \backslash X_2^+\big)$, then there exists $X_3 \in \mathcal{C}$ with $f \in X_3 \subseteq (X_1 \cup X_2)\backslash\{e\}$; and*

(ii) *there are $e_1 \in X_1 \backslash X_2$, $e_2 \in X_2 \backslash X_1$, and $X_4 \in \mathcal{C}$ satisfying $X_4 \subseteq (X_1 \cup X_2)\backslash\{e\}$ so that $X_4(e_1)X_4(e_2) = X_1(e_1)X_2(e_2)$.*

As with orientations, weak orientations can be characterized by forbidden minors.

Theorem 2.4 ([8, Theorem 1, p. 173])**.** *A circuit signature $\mathcal{C}$ of a matroid is a weak orientation of M if and only if $\mathcal{C}$ has no minor isomorphic to a reorientation of the signature $\{12, 13, 23\}$ of $U_{1,3}$.*

2.6. **Ternary Signatures of Matroids.** In order to introduce ternary signatures, we first discuss chain groups and their associated matroids. This theory was developed by Tutte. We use terminology, notation, and theorems from [14, Chapter 8] and [15, Section 9.4].

2.6.1. *Chain-Group Matroids.* Let F be a field, and let S be a finite set. A *chain on S to F* is a map $f : S \to F$. Denote the collection of all chains on S to F by F^S. The *support* of a chain f, denoted by $||f||$, is the set $\{e \in S : f(e) \neq 0\}$.

A *chain group on S to F* is a subspace of F^S.

Let N be a chain group on S to F. A chain f is *elementary* if its support is nonempty and, for all nonzero chains g of N with $||g|| \subseteq ||f||$, we have $||g|| = ||f||$.

Given a chain group N on S to F, there is a corresponding matroid $M(N)$ called the *chain-group matroid* of N. The circuits of $M(N)$ are the supports of the elementary chains of N (see [15, Section 9.4]). (Theorem 1 in [15, Section 9.4] is false. It incorrectly describes the dependent sets of $M(N)$. The proof, however, concentrates on describing the circuits of $M(N)$, and this description is correct.)

The class of chain-group matroids on S to F is exactly the class of matroids on S which are linearly representable over F. We sketch a part of the proof for later reference.

Theorem 2.5 ([15, Section 9.4, Theorem 2])**.** *A matroid M is isomorphic to the chain-group matroid of a chain group over a field F if and only if M is representable over F.*

Sketch of the proof of sufficiency. Let M be a matroid on S, and let $\phi : S \to V$ represent M in V, a vector space over F. Then $M \cong M(N)$ where N is the kernel of the map $\alpha : F^S \to V$ defined by

$$\alpha(f) = \sum_{e \in S} f(e)\phi(e).$$

□

2.6.2. *Ternary Signatures.* Now we use Tutte's theory of chain groups to construct circuit signatures for ternary matroids, signatures which we later call ternary signatures. We first need Lemma 2.7, whose proof depends on the following result.

Lemma 2.6 ([11, Proposition 2.2.23])**.** *Let $[I_r \mid D]$ be an $r \times n$ matrix over a field F where $1 \leq r \leq n-1$. Then the orthogonal complement in F^n of the row space of $[I_r \mid D]$ is the row space of $\big[-D^T \mid I_{n-r}\big]$.*

Let M be a matroid that is representable over F. This means that there exists a *representation matrix* whose entries are in F and whose columns are labeled by the elements of E such that a set is independent in M if and only if the associated columns of the matrix are linearly independent.

Lemma 2.7. *Let M be a matroid that is representable over F. Let N be the row space of a representation matrix of M^*. Then $M \cong M(N)$.*

Proof. Assume that M is a rank r matroid on a set of size n. Let A be a representation matrix of M^*, and let N be the row space of A. We may assume that the last $n-r$ columns of A are linearly independent. (Otherwise, we could permute columns.) Thus we can apply elementary row operations to reduce A to the matrix $[D \mid I_{n-r}]$, whose row space is also N. It follows that $\big[I_r \mid -D^T\big]$ is a representation matrix of M over F $\big($see [11, Theorem 2.2.8]$\big)$. Denote the columns of this matrix by $\phi(s_1), \ldots, \phi(s_n)$.

We need to show that $M \cong M(N)$. From the sketch of the proof of Theorem 2.5, it suffices to show that the row space of $[D \mid I_{n-r}]$ is the kernel of the map $\alpha : F^S \to V$ defined by

$$\alpha\big((g_1, \ldots, g_n)\big) = \sum_{i=1}^{n} g_i \phi(s_i).$$

Assume that $1 \leq r \leq n-1$. By applying Lemma 2.6, we can instead prove that the orthogonal complement of the row space of $\big[I_r \mid -D^T\big]$ is the kernel of α.

Let $(g_1, \ldots, g_n)$ be in the orthogonal complement of the row space of $\big[I_r \mid -D^T\big]$. By definition, $\alpha\big((g_1, \ldots, g_n)\big)$ is an $n \times 1$ column vector whose ith component is the dot product of $(g_1, \ldots, g_n)$ and the ith row of $\big[I_r \mid -D^T\big]$. Thus $\alpha\big((g_1, \ldots, g_n)\big) = 0$. Equivalently, $(g_1, \ldots, g_n) \in \ker \alpha$.

If $r = n$, then $M \cong U_{n,n}$ and $M^* \cong U_{0,n}$. The row space of any representation of M^* consists only of 0, and M can be represented by I_n. Thus α is the identity map, so $\ker \alpha = \{0\}$.

The proof for $r = 0$ is similar to the proof for $r = n$. □

Recall that the circuits of M are the supports of the elementary vectors of N.

Now we restrict Lemma 2.7 to the case in which $F = \mathrm{GF}(3)$. This is discussed by Roudneff and Wagowski in [12]. (They call N the *Tutte representation* of M over $GF(3)$.) The elementary vectors of N can be used to obtain a circuit signature of M: an element s in a signed circuit is positive if the value in the s-coordinate of the elementary vector is $+1$, and it is negative otherwise. This circuit signature is called a *ternary signature* of M.

Example 2.8. As an example, consider the representation

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

of $U_{2,4}$ over GF(3). The elementary vectors of the row space of this representation are

$$(1 \quad 1 \quad -1 \quad 0), \quad (1 \quad -1 \quad 0 \quad -1), \quad (1 \quad 0 \quad 1 \quad 1), \quad (0 \quad 1 \quad 1 \quad -1),$$

and their negatives. This gives the ternary signature $\big\{12\overline{3}, 1\overline{2}\overline{4}, 134, 23\overline{4}\big\}$ of $U_{2,4}$.

2.6.3. *The Uniqueness of Ternary Signatures.* In Example 2.8, it initially appears that a ternary signature of M depends on the representation matrix of M^*. However, Corollary 2.10 says that a ternary matroid has a "unique" representation over GF(3) (the word "unique" will be defined shortly). Theorem 2.11 shows that the effect of this uniqueness is that the ternary signature is unique up to reorientation. Though the uniqueness of ternary signatures is certainly known, we include a proof due to the lack of a reference where this result is explicitly stated.

Certainly, a representation matrix of a matroid cannot be unique. Multiplying any column by a scalar is a different representation matrix, strictly speaking. So we must define what it means for a matroid to be uniquely F-representable. We use the terminology of Brylawski and Lucas that is found in [7, Sections 2 and 3].

Let M be a rank r matroid with cardinality n that is representable over a field F, and let A_1 and A_2 be representation matrices of M. We define A_1 and A_2 to be *projectively equivalent* if A_2 can be obtained from A_1 by a sequence of operations of the five types listed below.

(1) Add a scalar multiple of one row to another.
(2) Interchange two rows.
(3) Multiply a row by a non-zero member of F.
(4) Remove or add a zero row.
(5) Multiply a column by a non-zero member of F.

We say that M is *uniquely F-representable* if it can be represented by an $r \times n$ matrix over F and all such matrices are projectively equivalent.

Lemma 2.9 ([7, Theorem 3.2]). *Let M be a matroid and let $[I \mid A_1]$ and $[I \mid A_2]$ be two representation matrices of M over a field F such that every entry of A_1 and A_2 is* 0, 1, *or* -1. *Then $[I \mid A_1]$ and $[I \mid A_2]$ are projectively equivalent.*

We easily get the following corollary to Lemma 2.9.

Corollary 2.10. *Ternary matroids are uniquely GF(3)-representable.*

Theorem 2.11. *A ternary matroid has a unique ternary signature, up to reorientation.*

Proof. Let M be a rank r ternary matroid on n elements. By definition, a ternary signature is determined by the row space of a representation matrix of M^*. Of course, M^* is ternary too. Let A_1 and A_2 be GF(3)-representations of M^*. By Corollary 2.10, A_1 and A_2 are projectively equivalent.

If A_1 and A_2 differ by row operations of types (1)–(4), they have the same row space, so the corresponding ternary signatures of M are the same.

Suppose that A_1 and A_2 differ by an operation of type (5), and assume that this column is labeled by element e. We may assume that A_1 and A_2 have r rows; otherwise we could apply row operations of types (1)–(4) to get matrices that have r rows and are row equivalent to A_1 and A_2. Since the field is GF(3), we need only be concerned about when the columns of A_1 and A_2 that are labeled by e are negatives of each other. Suppose $\{s_1, \ldots, s_k\}$ is a circuit of the chain-group matroid M(row space of A_1). So this row space contains some elementary chain, say

$$v = \alpha_1 \cdot (\text{row } 1 \text{ of } A_1) + \cdots + \alpha_r \cdot (\text{row } r \text{ of } A_1),$$

whose support is $\{s_1, \ldots, s_k\}$. The vector

$$w = \alpha_1 \cdot (\text{row } 1 \text{ of } A_2) + \cdots + \alpha_r \cdot (\text{row } r \text{ of } A_2)$$

differs from v only in the coordinate labeled by e, and the entry in this coordinate differs by a multiplier of -1. So w has support $\{s_1, \ldots, s_k\}$ also. Moreover, $\{s_1, \ldots, s_k\}$ is a circuit in M(row space of A_2) because w must be elementary. Thus the chain-group matroids M(row space of A_1) and M(row space of A_2) are identical, and the corresponding ternary signatures of M differ by a reorientation of e.

The complete proof follows by induction on the number of operations of types (1)–(5) by which A_1 and A_2 differ. $\square$

2.6.4. *Characterizing Ternary Signatures.* Since we are interested in circuit signatures, it is convenient that Roudneff and Wagowski characterize ternary signatures by a signed circuit elimination axiom [12, Theorem 3.1].

Theorem 2.12. *Let M be a matroid and $\mathcal{C}$ a signature of its circuits. Then the following properties are equivalent:*

(1) *$\mathcal{C}$ is a ternary signature.*
(2) *For any $X_1, X_2 \in \mathcal{C}$ with $\left(X_1^+ \cap X_2^-\right) \cup \left(X_1^- \cap X_2^+\right) \neq \emptyset$ and for any $f \in \left(X_1^+ \backslash X_2^-\right) \cup \left(X_1^- \backslash X_2^+\right)$, there exists $X_3 \in \mathcal{C}$ such that $f \in X_3 \subseteq (X_1 \cup X_2) \backslash \left((X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)\right)$, and there exist $e_1 \in X_1 \cap X_3$ and $e_2 \in X_2 \cap X_3$ such that $X_1(e_1)X_2(e_2) = X_3(e_1)X_3(e_2)$.*
(3) *There exists a signature $\mathcal{C}^*$ of the cocircuits of M such that for any $X \in \mathcal{C}$ and any $Y \in \mathcal{C}^*$ with $|X \cap Y| = 2, 3$, we have $\left|(X^+ \cap Y^+) \cup (X^- \cap Y^-)\right| \equiv \left|(X^+ \cap Y^-) \cup (X^- \cup Y^+)\right| \mod 3$.*
(4) *$\mathcal{C}$ has no minor isomorphic to a reorientation of the circuit signature $\{12, 13, 23\}$ of $U_{1,3}$, or to a reorientation of the circuit signature $\left\{123, 1\overline{2}4, 134, 23\overline{4}\right\}$ of $U_{2,4}$.*

3. How to Characterize Weak Orientations, Orientations, and Ternary Signatures by Modular Triples

Weak orientations, orientations, and ternary signatures are characterized in the literature in a variety of ways. In Section 2, we mentioned characterizations by forbidden minors, circuit elimination, and orthogonality. We provide a new characterization of these circuit signatures, as well as of lifting signatures (which are defined in Section 4.1), in terms of modular triples of circuits. These characterizations appear in Theorem 3.1. I first discovered this theorem for lifting signatures, a context in which modular triples of circuits naturally appear (the word "naturally" is explained in Section 4.1). Later, I realized that the theorem applies to the other signatures as well.

Recall that the *exponent* of a group $\mathfrak{G}$, denoted by $\exp(\mathfrak{G})$, is the smallest positive integer e (if it exists) such that $g^e = 1$ for all $g \in \mathfrak{G}$. If no such integer exists, then $\exp(\mathfrak{G}) = \infty$.

Let $\mathcal{C}$ be a circuit signature of a matroid M. In Theorem 3.1, we refer to the following property, which we call the *Well-Distribution Property* (WDP): For each modular triple of signed circuits, (C_1, C_2, C_3), there exist sets I_1, I_2, I_3, and I_4 with $I_1 \mathbin{\dot\cup} I_2 = I_3 \mathbin{\dot\cup} I_4 = I$ so that, up to reorientation,

$$C_1 = (I \cup I_{13}, I_{12}),$$

$$C_2 = \pm(I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$

$$C_3 = \pm(I_3 \cup I_{23}, I_4 \cup I_{13}).$$

(The sets I, I_{13}, I_{12}, and I_{23} are defined in Figure 2.1.)

Theorem 3.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M.*

(1) *$\mathcal{C}$ is a weak orientation of M if and only if the Well-Distribution Property holds.*
(2) *$\mathcal{C}$ is an orientation of M if and only if the Well-Distribution Property holds with $I_3 \subseteq I_2$.*
(3) *$\mathcal{C}$ is a ternary signature of M if and only if the Well-Distribution Property holds with $I_1 = I_3 = I$.*
(4) *Let $\mathfrak{A}$ be an abelian group with* $\exp(\mathfrak{A}) > 2$.
    (a) *Assume M is binary. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if the Well-Distribution Property holds with $I_1 = I_3 = I$.*
    (b) *Assume M is not binary. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if* $\exp(\mathfrak{A}) = 3$ *and the Well-Distribution Property holds with $I_1 = I_3 = I$.*

In this section, we prove Theorem 3.1(1)–(3); part (4) is proved in Section 4.2. When proving Theorem 3.1(1)–(3), we use the fact that if $\mathcal{C}$ is an orientation or ternary signature of M, then $\mathcal{C}$ is a weak orientation of M.

Since orientations and ternary signatures are weak orientations, we can use Theorem 3.1(1) to help with the proofs of necessity of parts (2) and (3). For the sake of variety, we use two different techniques to prove the sufficiency of these three results. To prove the sufficiency of part (2), we use Las Vergnas' result that signed circuit elimination for orientations is equivalent to modular signed circuit elimination. To prove the sufficiency of parts (1) and (3), we use forbidden minor arguments. It is possible, however, to prove the sufficiency of all three results using either one of these methods.

The forbidden minor arguments require Lemmas 3.2 and 3.3.

Lemma 3.2. *Let M be a matroid, and let a be an element of M. Let (X_1', X_2') be a modular pair of circuits in M/a. Then the (unique) circuits X_1 and X_2 of M such that $X_1' = X_1 \backslash \{a\}$ and $X_2' = X_2 \backslash \{a\}$ constitute a modular pair of circuits in M.*

Proof. See the proof of Lemma 2.3 in [10]. □

In addition to the WDP, other requirements involving the sets I_1, I_2, and I_3 appear in Theorem 3.1(2)–(4). Let Property P be the WDP together with one (or none) of these additional requirements.

Lemma 3.3. *Let $\mathcal{C}$ be a circuit signature of a matroid M that satisfies Property P. Then any minor of $\mathcal{C}$ also satisfies Property P.*

Proof. We show that if $\mathcal{C}$ satisfies Property P, then $\mathcal{C}\backslash s$ and $\mathcal{C}/s$ also satisfy Property P. We prove the contrapositive of this statement by induction on $|E(M)|$. The base case, when $|E(M)| = 1$, is vacuously true because M has at most one circuit.

Assume that $|E(M)| > 1$. If $\mathcal{C}\backslash s$ does not satisfy Property P, then $\mathcal{C}\backslash s$ has a modular triple of signed circuits, say (C_1, C_2, C_3), that does not satisfy Property P. But (C_1, C_2, C_3) is also a modular triple of circuits of M, and the signatures of C_i in $\mathcal{C}$ and $\mathcal{C}\backslash s$ are the same. Therefore, $\mathcal{C}$ does not satisfy Property P either.

If $\mathcal{C}/s$ does not satisfy Property P, then $\mathcal{C}/s$ has a modular triple of signed circuits, say (C_1, C_2, C_3), that does not satisfy Property P. Ignoring signatures momentarily, we claim that M has a modular triple of circuits, (D_1, D_2, D_3), such that $C_i \subseteq D_i$. The signature of D_i is an extension of the signature of C_i, so (D_1, D_2, D_3) cannot satisfy Property P because this would imply that (C_1, C_2, C_3) satisfies Property P.

To prove our claim, we apply Lemma 3.2 and see that M has circuits D_1, D_2, and D_3 such that $C_i = D_i\backslash\{s\}$ and (D_1, D_2), (D_1, D_3), and (D_2, D_3) are modular pairs of circuits. We know that $D_i = C_i$ or $D_i = C_i \cup \{s\}$. Also, D_1, D_2, and D_3 are distinct circuits because C_1, C_2, and C_3 are distinct. To prove that (D_1, D_2, D_3) is a modular triple of circuits of M, we need only prove that $D_i \subseteq (D_j \cup D_k)$ for distinct i, j, and k. The only way that $D_i \subseteq (D_j \cup D_k)$ can fail is when $D_i = C_i \cup \{s\}$, $D_j = C_j$, and $D_k = C_k$. Suppose this happens. Let E be the ground set of M. Define $H_i^* = \big(E\backslash\{s\}\big)\backslash C_i$. Then (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of $M^*\backslash s$. Also, $\big(H_i^*, H_j^* \cup \{s\}, H_k^* \cup \{s\}\big)$ is a modular triple of copoints of M^*. Since M^* is a single-element extension of $M^*\backslash s$, we can tell from Figure 2.2 (see Section 2.1) that H_j^* and H_k^* are in the associated modular cut of $M^*\backslash s$, but H_i^* is not in the modular cut. According to the definition of a modular cut, this is impossible because (H_j^*, H_k^*) is a modular pair and so H_i^* must be in the modular cut as well. This contradiction proves that $D_i \subseteq (D_j \cup D_k)$, which concludes the proof of our claim.

A complete proof of the lemma follows by induction on the number of deletions and contractions that are required to obtain the minor of $\mathcal{C}$. $\square$

Lemmas 3.4 and 3.5 help prove Theorem 3.1(1)–(3). Lemma 3.4 allows the freedom to reorient circuit signatures without changing their type, and Lemma 3.5 is an easy technical result.

Lemma 3.4. *Let $\mathcal{C}$ be a circuit signature of M, and let $A \subseteq E$.*

(1) [1, Proposition 1.7] *If $\mathcal{C}$ is a weak orientation of M, then ${}_{\overline{A}}\mathcal{C}$ is also a weak orientation of M.*
(2) [2, Section 2] *If $\mathcal{C}$ is an orientation of M, then ${}_{\overline{A}}\mathcal{C}$ is also an orientation of M.*
(3) [12, Section 2] *If $\mathcal{C}$ is a ternary signature of M, then ${}_{\overline{A}}\mathcal{C}$ is also a ternary signature of M.*

Lemma 3.5. *Let $\mathcal{C}$ be a weak orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. Let i, j, and k be distinct elements of $\{1, 2, 3\}$. If $\{x_1, x_2\} \subseteq (C_i \cap C_j)\backslash C_k$ and $x_1 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$, then $x_2 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$.*

Proof. We may assume that $x_1 \in C_i^+ \cap C_j^+$. Otherwise, we could proceed with the proof using $-C_i$ and $-C_j$. Assume the conclusion is false, and relabel, if necessary, so that $x_2 \in C_i^+ \cap C_j^-$. By Theorem 2.3(i), there exists $X_3 \in \mathcal{C}$ with $x_1 \in X_3 \subseteq (C_i \cup C_j)\backslash\{x_2\}$. By Lemma 2.2, there is a unique circuit contained in $(C_i \cup C_j)\backslash\{x_2\}$, namely C_k. Thus $X_3 = \pm C_k$, a contradiction because $x_1 \notin C_k$. □

When proving Theorem 3.1(1)–(3), we frequently simplify notation by making assumptions about reorientation and negation. These assumptions will be explained in the proof of part (1), but not thereafter.

Proof of Theorem 3.1(1). Assume $\mathcal{C}$ is a weak orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Lemma 3.4(1), we may assume that

$$C_1 = (I \cup I_{13}, I_{12}).$$

(Technically, we reorient by A so that ${}_{\overline{A}}C_1 = (I \cup I_{13}, I_{12})$. The structure of the signed circuits in the WDP, however, is specified only up to reorientation, so there is no need to complicate the argument with additional notation.)

We show that either $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq C_2^-$. If not, there exist y_1 and y_2, both elements of I_{12}, such that $y_1 \in C_2^+ \cap C_1^-$ and $y_2 \in C_2^- \cap C_1^-$. This contradicts Lemma 3.5. We may assume that $I_{12} \subseteq C_2^+$. (Technically, $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq (-C_2)^+$. The structure of C_2 in the WDP, however, is specified only up to negation. By reorientation in I_{23}, we may also assume that $I_{23} \subseteq C_2^-$. (This is possible since I_{23} is disjoint from C_1.) We have found that

$$C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$$

where $I_1 \mathbin{\dot\cup} I_2 = I$.

Notational adjustments to the argument above prove that $I_{13} \subseteq C_3^+$ or $I_{13} \subseteq C_3^-$, and that $I_{23} \subseteq C_3^+$ or $I_{23} \subseteq C_3^-$. Suppose that the elements of $I_{13} \cup I_{23}$ all have the same sign in C_3. We may assume that $I_{13} \cup I_{23} \subseteq C_3^-$. Choose $x \in I_{13}$. When we apply Theorem 2.3(ii) to x, C_1, and C_3, we find that $e_1 \in I_{12}$ and $e_2 \in I_{23}$, so $C_1(e_1)C_3(e_2) = (-1)(-1) = +1$. However, by Lemma 2.2, $X_4 = \pm C_2$, and in both cases $X_4(e_1)X_4(e_2) = -1$, a contradiction. Thus,

$$C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$$

where $I_3 \mathbin{\dot\cup} I_4 = I$.

Now assume that $\mathcal{C}$ satisfies the WDP. By Lemma 3.3, any minor of $\mathcal{C}$ must also satisfy the WDP. Thus an induced $U_{1,3}$ circuit signature must be isomorphic to a reorientation of $\{12, 13, 2\overline{3}\}$. Using Theorem 2.4, we conclude that $\mathcal{C}$ is a weak orientation of M. □

Proof of Theorem 3.1(2). Assume that $\mathcal{C}$ is an orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Theorem 3.1(1) and Lemma 3.4(2) (and possibly replacing C_2 or C_3 with their negatives), we may assume that

$$C_1 = (I \cup I_{13}, I_{12}),$$
$$C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$
$$C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13}),$$

where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$.

Choose $y \in I_{12}$. By applying signed circuit elimination to y, C_1, and C_2, we find that $\mathcal{C}$ has a signed circuit $C \subseteq (C_1 \cup C_2)\backslash\{y\}$ with

$$C^+ \subseteq I \cup I_{13} \cup I_{12}\backslash\{y\} \text{ and } C^- \subseteq I_2 \cup I_{12}\backslash\{y\} \cup I_{23}.$$

By Lemma 2.2, $C = \pm C_3$. Thus $(I_3 \cup I_{23}) \subseteq (I_2 \cup I_{23})$, which implies that $I_3 \subseteq I_2$.

Now assume that $\mathcal{C}$ satisfies the WDP with $I_3 \subseteq I_2$. To prove that $\mathcal{C}$ is an orientation of M, we need to prove the circuit elimination axiom for orientations. This axiom appears in Section 2.4. In that section, we also mention that circuit elimination only needs to be verified for modular pairs of signed circuits. Let (C_1, C_2) be a modular pair of signed circuits such that $C_1 \neq \pm C_2$, and let $e \in C_1^+ \cap C_2^-$. By Lemma 2.2, there exists a unique circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2)\backslash\{e\}$. Moreover, (C_1, C_2, C_3) is a modular triple. According to our notation, we know that $e \in I_{12}$.

We must prove that for $\tau = +$ or $\tau = -$, $(\tau C_3)^+ \subseteq (C_1^+ \cup C_2^+)\backslash\{e\}$ and $(\tau C_3)^- \subseteq (C_1^- \cup C_2^-)\backslash\{e\}$. Up to reorientation, we know that

$$C_1 = (I \cup I_{13}, I_{12}),$$
$$C_2 = \pm(I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$
$$C_3 = \pm(I_3 \cup I_{23}, I_4 \cup I_{13}),$$

where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$ and $I_3 \subseteq I_2$. By construction, e has apposite signs in C_1 and C_2 (both before and after reorientation). Thus $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$. Since we assumed that $I_3 \subseteq I_2$, it follows that

$$(\tau C_3)^+ \subseteq C_1^+ \cup C_2^+ \text{ and } (\tau C_3)^- \subseteq C_1^- \cup C_2^-$$

for $\tau = +$ or $\tau = -$. □

Proof of Theorem 3.1(3). Assume that $\mathcal{C}$ is a ternary signature, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Theorem 3.1(1) and Lemma 3.4(3) (and possibly replacing C_2 or C_3 with their negatives), we may assume that

$$\begin{aligned} C_1 &= (I \cup I_{13}, I_{12}), \\ C_2 &= (I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and} \\ C_3 &= (I_3 \cup I_{23}, I_4 \cup I_{13}), \end{aligned}$$

where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$.

Choose $x \in I_{13}$, and let C_1, C_2, and x play the respective roles of X_1, X_2, and f in Theorem 2.12(2). Thus there exists $X_3 \in \mathcal{C}$ such that $X_3 \subseteq (C_1 \cup C_2) \backslash (I_2 \cup I_{12})$. But C_3 and $-C_3$ are the only signed circuits contained in $(C_1 \cup C_2) \backslash I_{12}$, so $X_3 = \pm C_3$. However, $I \subseteq C_3$, so $I_2 = \emptyset$.

An identical argument shows that $I_4 = \emptyset$, which concludes the proof of necessity.

To prove sufficiency, assume that $\mathcal{C}$ satisfies the WDP with $I_1 = I_3 = I$. By Theorems 3.1(1) and 2.4, $\mathcal{C}$ has no minor isomorphic to a reorientation of $\{12, 13, 23\}$.

According to Lemma 3.3, any minor that is a signature of $U_{2,4}$ must also satisfy the WDP with $I_1 = I_3 = I$. We claim that such a minor is a reorientation of $\left\{12\overline{3}, 1\overline{2}4, 13\overline{4}, 234\right\}$. There is no way to reorient this signature so that exactly two circuits have positive signatures, thus

$\mathcal{C}$ has no minor isomorphic to a reorientation of the signatures in Theorem 2.12(4). It follows that $\mathcal{C}$ is a ternary signature of M.

We conclude with a proof of our claim. Since any three circuits of $U_{2,4}$ form a modular triple, we know that the signatures of $\{1,2,3\}$, $\{1,2,4\}$, and $\{1,3,4\}$ are some reorientation of $12\overline{3}$, $1\overline{2}4$, and $13\overline{4}$. We also know that the signatures of $\{1,2,3\}$, $\{1,2,4\}$, and $\{2,3,4\}$ are some reorientation of $12\overline{3}$, $\overline{1}24$, and $23\overline{4}$. Putting these two facts together, we see that the circuit signature of $U_{2,4}$ must be some reorientation of $\big\{12\overline{3}, 1\overline{2}4, 13\overline{4}, 234\big\}$. □

4. LIFTING SIGNATURES

4.1. **Definitions.** Now we generalize Section 2.2, where gains enabled the construction of graphic-matroid lifts. The main idea is to replace information obtained from graphs with information obtained from matroid circuit signatures.

Let $\Phi = (\Gamma, \phi)$ be a gain graph with gain group $\mathfrak{G}$. We can think of Γ as a directed graph because ϕ oriented the edges in order to assign gains. There is a standard way of associating this directed graph with an orientation $\mathcal{C}$ of the graphic matroid $G(\Gamma)$ $\big($see [4, Section 1.1]$\big)$. Arbitrarily assign an orientation to each circle of Γ; an element of a signed circuit is positive if its direction agrees with the orientation assigned that circle, and it is negative otherwise.

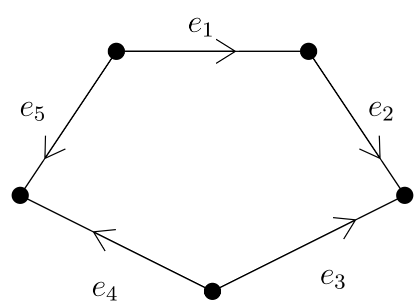


FIGURE 4.1. This is a circle of Φ. The arrows on the edges indicate the orientations prescribed by ϕ.

Suppose the circle B in Figure 4.1 is in Γ. According to Section 2.2, B is balanced if and only if

$$\phi(e_1)\phi(e_2)\phi(e_3)^{-1}\phi(e_4)\phi(e_5)^{-1} = 1.$$

Balance can also be defined using the circuit signature $\mathcal{C}$ we described above. In our example,

$$\big(\{e_1, e_2, e_4\}, \{e_3, e_5\}\big) \in \mathcal{C}.$$

Assuming that the gain group is abelian, B is balanced if and only if

$$\prod_{e \in B^+} \phi(e) \prod_{e \in B^-} \phi(e)^{-1} = 1.$$

We require that the gain group be abelian; otherwise, this product may not be well defined.

Our example illustrates how the circuit signature that is associated with a gain graph provides a different way of determining which circles are balanced. Moreover, this method lends itself to a matroid generalization, where circuit signatures are used to determine whether or not a matroid circuit is balanced. Now we give the formal definitions that are necessary for this generalization.

Let M be a matroid on E, let $\mathcal{C}$ be a circuit signature of M, and let $\mathfrak{A}$ be an abelian group. A *gain mapping* ϕ is a function from E into $\mathfrak{A}$. We call $\mathfrak{A}$ the *gain group.*

Let C be a circuit of M, so C is the support of two signed circuits in $\mathcal{C}$. Suppose one of these signed circuits is $\big(\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_n\}\big)$. We define the *gain* of C to be

$$\phi(C) = \prod_{a \in C^+} \phi(a) \prod_{b \in C^-} \phi(b)^{-1}.$$

We say that C is *balanced* if $\phi(C) = 1$. Whether or not C is balanced is independent of which of the two signed circuits with support C is used for the computation. Let $\mathcal{B}(\phi, \mathcal{C})$ denote the class of balanced circuits. If $\mathcal{C}$ is clear from context, we write $\mathcal{B}(\phi)$. If $\mathcal{B}(\phi)$ is a linear class of circuits, we can apply Dowling and Kelly's lift construction (as in Section 2.1) to obtain $L\big(M, \mathcal{B}(\phi)\big)$, an elementary lift of M.

It is certainly not the case that $\mathcal{B}(\phi, \mathcal{C})$ is linear for all choices of ϕ and $\mathcal{C}$. For example, consider the matroid $U_{2,4}$ with orientation

$$\big\{1\overline{2}3, 1\overline{2}4, 1\overline{3}4, 2\overline{3}4\big\}. \tag{4.1}$$

Let the gain group be $\mathbb{R}^+$, and define ϕ by

$$\phi(1) = 1, \quad \phi(2) = 0, \quad \text{and} \quad \phi(3) = \phi(4) = -1.$$

Then $\phi(123) = \phi(124) = 0$, but $\phi(134) = 1$. Thus $\mathcal{B}(\phi, \mathcal{C})$ is not linear, and the lift construction cannot be applied. There certainly exists a gain mapping τ that makes $\mathcal{B}(\tau, \mathcal{C})$ linear. However, we want to generalize Section 2.2 (the graphical case), where $\mathcal{B}(\phi)$ is always linear. Thus the $U_{2,4}$ example teaches us that we must be more selective when choosing the circuit signature.

The discussion above inspires the following definitions. A matroid M can be *lifted by gains in* $\mathfrak{A}$ if M has a circuit signature $\mathcal{C}$ such that $\mathcal{B}(\phi, \mathcal{C})$ is linear for all $\phi : E \to \mathfrak{A}$. In this case, we call $\mathcal{C}$ a *lifting signature for gains in* $\mathfrak{A}$. For example, the orientation associated with a graph Γ is a lifting signature of the graphic matroid $G(\Gamma)$ (for gains in any group).

Now we ask, which matroids (in addition to graphic matroids) have lifting signatures? Since linear classes of circuits are central to the definition of a lifting signature and since they are defined in terms of modular triples, it is natural that modular triples be used to characterize lifting signatures.

4.2. **How to Characterize Lifting Signatures by Modular Triples.** In this section we prove Theorem 3.1(4), which characterizes lifting signatures in terms of modular triples of signed circuits. The following Theorem is then an immediate consequence of Theorem 3.1.

Theorem 4.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M, and let $\mathfrak{A}$ be an abelian group with $\exp(\mathfrak{A}) > 2$. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if $\mathcal{C}$ is a ternary signature and $\exp(\mathfrak{A}) = 3$ when M is not binary.*

Before proving Theorem 3.1(4), we prove two lemmas, which parallel the reorientation and technical lemmas that contributed to the proofs of Theorem 3.1(1)–(3).

Given a gain mapping ϕ, define a new gain mapping ϕ_A by

$$\phi_A(e) = \begin{cases} \phi(e) & \text{if } e \notin A, \\ \phi(e)^{-1} & \text{if } e \in A. \end{cases}$$

Lemma 4.2. *Let $\mathcal{C}$ be a circuit signature of M, let $A \subseteq E$, and let $\mathfrak{A}$ be an abelian group.*

(1) *For each gain mapping ϕ, $\mathcal{B}(\phi_A, {}_{\overline{A}}\mathcal{C}) = \mathcal{B}(\phi, \mathcal{C})$.*
(2) *$\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if ${}_{\overline{A}}\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$.*

Proof. Let ϕ be a gain mapping, and let C be a circuit of M. Throughout this proof, $\phi(C)$ is calculated using $\mathcal{C}$ and $\phi_A(C)$ is calculated using ${}_{\overline{A}}\mathcal{C}$.

Assume that C is the support of the signed circuit

$$\big(\{p_1, \ldots, p_r, a_1, \ldots, a_s\}, \{n_1, \ldots, n_t, b_1, \ldots, b_q\}\big)$$

of $\mathcal{C}$, where $A \cap C = \{a_1, \ldots, a_s, b_1, \ldots, b_q\}$. Then C is the support of the signed circuit

$$\big(\{p_1, \ldots, p_r, b_1, \ldots, b_q\}, \{n_1, \ldots, n_t, a_1, \ldots, a_s\}\big)$$

of ${}_{\overline{A}}\mathcal{C}$. Accordingly,

$$\begin{aligned}\phi_A(C) &= \prod_{i=1}^{r}\phi_A(p_i)\prod_{i=1}^{q}\phi_A(b_i)\prod_{i=1}^{t}\phi_A(n_i)^{-1}\prod_{i=1}^{s}\phi_A(a_i)^{-1}\\ &= \prod_{i=1}^{r}\phi(p_i)\prod_{i=1}^{s}\phi(a_i)\prod_{i=1}^{t}\phi(n_i)^{-1}\prod_{i=1}^{q}\phi(b_i)^{-1}\\ &= \phi(C).\end{aligned}$$

It follows immediately that $\mathcal{B}(\phi_A, {}_{\overline{A}}\mathcal{C}) = \mathcal{B}(\phi, \mathcal{C})$.

To prove part (2), assume that $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. So there exist a gain mapping ϕ and a modular triple of circuits, (C_1, C_2, C_3), such that $\phi(C_1) = \phi(C_2) = 1$ and $\phi(C_3) \neq 1$. From part (1), it follows that $\phi_A(C_1) = \phi_A(C_2) = 1$ and $\phi_A(C_3) \neq 1$. Thus ${}_{\overline{A}}\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$.

Now assume that ${}_{\overline{A}}\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. We just proved that this implies that ${}_{\overline{A}}({}_{\overline{A}}\mathcal{C})$ is not a lifting signature for gains in $\mathfrak{A}$. But ${}_{\overline{A}}({}_{\overline{A}}\mathcal{C}) = \mathcal{C}$. $\square$

Lemma 4.3. *Let $\mathfrak{A}$ be an abelian group with $\exp(\mathfrak{A}) > 2$, let $\mathcal{C}$ be a lifting signature of M for gains in $\mathfrak{A}$, and let (C_1, C_2, C_3) be a modular triple of signed circuits. Let i, j, and k be distinct element of $\{1, 2, 3\}$.*

(1) *Assume $\{x_1, x_2\} \subseteq (C_i \cap C_j)\backslash C_k$. If $x_1 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$, then $x_2 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$.*
(2) *Assume x, y, and z are each in exactly two of C_i, C_j, and C_k. If $x \in C_i^+ \cap C_k^-$, $y \in C_j^+ \cap C_i^-$, $z \in C_j^-$, and $z \in C_k$, then $z \in C_k^+$.*
(3) *Assume $y \in (C_i \cap C_j)\backslash C_k$ and $w \in C_i \cap C_j \cap C_k$. If $y \in C_i^- \cap C_j^+$ and $w \in C_i^+$, then $w \in C_j^+$.*

Proof. Throughout this proof, let $g \in \mathfrak{A}$ have order greater than 2.

For part (1), we may assume that $x_1 \in (C_i^+ \cap C_j^+)$. Otherwise, we could proceed with the proof using the modular triple $(-C_i, -C_j, C_k)$. Suppose the conclusion is false. By relabeling if necessary, we can assume that $x_2 \in (C_i^+ \cap C_j^-)$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{x_1, x_2\}, \\ 1 & \text{otherwise.}n \end{cases}$$

Then $\phi(C_j) = g \cdot g^{-1} = 1$ and $\phi(C_k) = 1$, but $\phi(C_i) = g \cdot g = g^2 \neq 1$, which contradicts the assumption that $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$.

For part (2), suppose $z \in C_k^-$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{x, y, z\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_i) = \phi(C_j) = g \cdot g^{-1} = 1$, but $\phi(C_k) = g^{-1} \cdot g^{-1} = (g^{-1})^2 \neq 1$, a contradiction.

For part (3), suppose $w \in C_j^-$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{w, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_i) = \phi(C_j) = g \cdot g^{-1} = 1$. It is not known whether w is positive or negative in C_k, so $\phi(C_k) = g$ or $\phi(C_k) = g^{-1}$. In either case, $\phi(C_k) \neq 1$, a contradiction. □

Theorem 3.1(4) is divided into two parts, depending on whether or not the matroid is binary. The two parts are made necessary by Theorem 2.1, which states that in a binary matroid, the intersection of the three circuits in a modular triple is empty, but that a nonbinary matroid has a modular triple for which this is not the case.

Proof of Theorem 3.1(4). Assume $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Lemma 4.2(2) (reorientation), we may assume that

$$C_1 = (I \cup I_{13}, I_{12}).$$

We will show that either $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq C_2^-$. If not, there exist y_1 and y_2, both elements of I_{12}, such that $y_1 \in C_1^- \cap C_2^-$ and $y_2 \in C_1^- \cap C_2^+$. This contradicts Lemma 4.3(1). We may assume that $I_{12} \subseteq C_2^+$ and that $I_{23} \subseteq C_2^-$. Applying Lemma 4.3(3), the elements of I have the same sign in C_2 as the elements of I_{12}. Thus

$$C_2 = (I \cup I_{12}, I_{23}).$$

An argument similar to the one above proves that either $I_{13} \subseteq C_3^+$ or $I_{13} \subseteq C_3^-$. Furthermore, by Lemmas 4.3(2) and 4.3(3), we find that the elements of $I \cup I_{23}$ and those of I_{13} have opposite signs in C_3. (Use 4.3(2) for I_{23} and 4.3(3) for I.) Thus

$$C_3 = (I \cup I_{23}, I_{13}).$$

We have proved the necessity of part (4a).

To prove the necessity of part (4b), we must show that $\exp(\mathfrak{A}) = 3$. Suppose $\exp(\mathfrak{A}) \neq 3$, so there exists $g \in \mathfrak{A}$ such that $g^3 \neq 1$. Since M is not binary, we apply Lemma 2.1 to find a modular triple of signed circuits, (C_1, C_2, C_3), with nonempty intersection. We must show that $\mathcal{C}$ is not a lifting signature. By the above argument, we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = \pm(I \cup I_{12}, I_{23})$, and $C_3 = \pm(I \cup I_{23}, I_{13})$. Choose $w \in I$, $x \in I_{13}$, and $z \in I_{23}$, and define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e = x, \\ g^{-1} & \text{if } e = w \text{ or } z, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_1) = \phi(C_2) = 1$, but $\phi(C_3)$ is $(g^{-1})^3$ or g^3, neither of which is 1. Thus $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. This contradicts our hypothesis, so $\exp(\mathfrak{A}) = 3$.

Now we prove sufficiency. Let (C_1, C_2, C_3) be a modular triple of signed circuits. We must prove that $\mathcal{C}$ is a lifting signature. By reorientation (and possibly replacing one or both of C_2 and C_3 by their negatives), we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I \cup I_{12}, I_{23})$, and $C_3 = (I \cup I_{23}, I_{13})$.

Let ϕ be a gain mapping. We must show that if $\phi(C_1) = \phi(C_2) = 1$, then $\phi(C_3) = 1$. (The other two combinations follow by relabeling.) If

$$\phi(C_1) = \phi(C_2) = 1,$$

then

$$\prod_{w \in I} \phi(w) \prod_{x \in I_{13}} \phi(x) \prod_{y \in I_{12}} \phi(y)^{-1} = \prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y) \prod_{z \in I_{23}} \phi(z)^{-1} = 1.$$

Thus

$$\begin{aligned}
\phi(C_3) &= \prod_{w \in I} \phi(w) \prod_{z \in I_{23}} \phi(z) \prod_{x \in I_{13}} \phi(x)^{-1} \\
&= \left(\prod_{w \in I} (\phi(w)\right) \left(\prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y)\right) \left(\prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y)^{-1}\right) \\
&= \prod_{w \in I} \big(\phi(w)\big)^3.
\end{aligned}$$

If M is binary, then $I = \emptyset$ (see Lemma 2.1). If M is not binary, then $\exp(\mathfrak{A}) = 3$. In both cases, $\phi(C_3) = 1$. □

4.3. **Using Gains to Lift Binary and Ternary Matroids.** So far, all results about lifting signatures mandate that the gain group have exponent greater than 2. The root cause of this is Lemma 4.3, whose proof requires an element of order greater than 2. Gain groups of exponent 2 have a different effect on lifting signatures because $\phi(e) = \phi(e)^{-1}$ for all elements of the matroid. Thus all circuit signatures behave like the all-positive signature.

Theorem 4.4 classifies the matroids that can be lifted by gains from a group of exponent greater than 2. Theorem 4.5 is a classification for gain groups of exponent 2.

Theorem 4.4. *Let M be a matroid, and let $\mathfrak{A}$ be an abelian group such that $\exp(\mathfrak{A}) > 2$. Then M can be lifted by gains in $\mathfrak{A}$ if and only if M is ternary and $\exp(\mathfrak{A}) = 3$ when M is not binary. Moreover, the lifting signature is the ternary signature associated with M, which is unique up to reorientation.*

Proof. M can be lifted by gains in $\mathfrak{A}$ if and only if M has a lifting signature for gains in $\mathfrak{A}$, call it $\mathcal{C}$. From Theorem 4.1, we see that $\mathcal{C}$ is also a ternary signature. But M has a ternary signature if and only if M is ternary. Moreover, Theorem 2.11 guarantees that a ternary matroid has precisely one ternary signature, up to reorientation. □

Theorem 4.5. *Let $\mathfrak{A}$ be an abelian group with $\exp(\mathfrak{A}) = 2$, and let $\mathcal{C}$ be a circuit signature of a matroid M. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if M is binary.*

Proof. Assume that M is not binary. By Lemma 2.1, there exists a modular triple of signed circuits, (C_1, C_2, C_3), with nonempty intersection. Let $w \in I$ and $y \in I_{12}$, and let $g \in \mathfrak{A}$ be any element other than 1. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{w, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_1) = \phi(C_2) = 1$, but $\phi(C_3) = g \neq 1$. Thus $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$.

Now assume that M is binary. Let (C_1, C_2, C_3) be a modular triple of signed circuits, and let ϕ be a gain mapping. By Lemma 2.1, we know that $I = \emptyset$. Since $\exp(\mathfrak{A}) = 2$, $\phi(e) = \phi(e)^{-1}$ for all $e \in E$, so we may assume that $\mathcal{C}$ is the all-positive signature. For distinct i, j, and k, we must show that if $\phi(C_i) = \phi(C_j) = 1$, then $\phi(C_k) = 1$ as well. We show this for one case; the other cases are identical up to renaming sets. Assume

$$\phi(C_1) = \phi(C_2) = 1.$$

Thus

$$\prod_{x \in I_{13}} \phi(x) \prod_{y \in I_{12}} \phi(y) = \prod_{y \in I_{12}} \phi(y) \prod_{z \in I_{23}} \phi(z) = 1.$$

Then

$$\phi(C_3) = \prod_{x \in I_{13}} \phi(x) \prod_{x \in I_{23}} \phi(z) = \prod_{y \in I_{12}} (\phi(y))^2 = 1.$$

□

5. Applications

This section consists of a variety of applications of Theorem 3.1. First, we provide quick, easy proofs of Corollary 5.1, which is largely a collection of known facts about orientations, weak orientations, and ternary signatures. Following our proof of each part, we give a reference for a previously known alternative proof. Next, we prove that the Fano matroid is not orientable. This fact follows from Corollary 5.1(2). However, we apply Theorem 3.1(2) directly in order to give the reader a better understanding of this theorem. Lastly, we go through the mechanics of using gains to construct a lift of a particular matroid.

Corollary 5.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M, and let $\mathfrak{A}$ be an abelian group where* $\exp(\mathfrak{A}) > 2$*, and* $\exp(\mathfrak{A}) = 3$ *if M is not binary.*

(1) *Assume M is binary. The following are equivalent: $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$, $\mathcal{C}$ is an orientation, $\mathcal{C}$ is a weak orientation, and $\mathcal{C}$ is a ternary signature.*
(2) *A binary matroid is orientable if and only if it is regular.*
(3) *Assume M is regular and $\mathcal{C}$ is an orientation. Then, up to reorientation, $\mathcal{C}$ is unique.*

(4) *If M is not binary and $\mathcal{C}$ is a ternary signature, then $\mathcal{C}$ is a weak orientation but is not an orientation.*

Proof. (1) By Lemma 2.1, if (C_1, C_2, C_3) is a modular triple of circuits of M, then $C_1 \cap C_2 \cap C_3 = \emptyset$. The proof now follows immediately from Theorem 3.1. (Previous proof: Combine [2, Theorem 2.2], [1, Theorem 1.10], 2.12(4), and 2.4. Of course, the inclusion of lifting signatures in this result is new.)

(2) A binary matroid M is orientable if and only if it has an orientation $\mathcal{C}$. By part (1), this is equivalent to $\mathcal{C}$ being a ternary signature. Hence M is ternary and binary, and therefore M is regular. (Previous proof: See [1, Proposition 7.9.3].)

(3) M is both binary and ternary. Since it is binary, part (1) says that $\mathcal{C}$ is also a ternary signature. But by Theorem 2.11, $\mathcal{C}$ is unique up to reorientation. (Previous proof: See [1, Corollary 7.9.4].)

(4) Since M is not binary, Lemma 2.1 guarantees the existence of a modular triple of circuits, (C_1, C_2, C_3), such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$. Thus I in Theorem 3.1 is nonempty. Since $\mathcal{C}$ is ternary, Theorem 3.1 indicates that, up to reorientation and negation,

$$C_1 = (I \cup I_{13}, I_{12}),$$

$$C_2 = (I \cup I_{12}, I_{23}), \text{ and}$$

$$C_3 = (I \cup I_{23}, I_{13}).$$

If $\mathcal{C}$ is also an orientation, then, up to reorientation and negation,

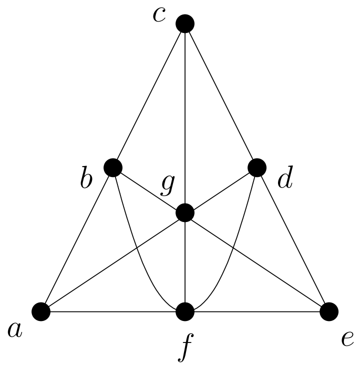
$$C_1 = (I \cup I_{13}, I_{12}),$$

$$C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23}), \text{ and}$$

$$C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$$

for some $I_1 \uplus I_2 = I_3 \uplus I_4 = I$ with $I_3 \subseteq I_2$. Thus $I_1 = I_3 = I$, and $I_2 = I_4 = \emptyset$. This contradicts $I_3 \subseteq I_2$. The result follows because ternary signatures are weak orientations. (Previous proof: See [12, Theorem 4.3].) $\square$

FIGURE 5.1. A geometric representation of F_7.



Theorem 5.2. *The Fano matroid $F_7 = PG(2, 2)$ is not orientable.*

Proof. Any two three-point circuits, C_1 and C_2, together with $C_1 \triangle C_2$, are a modular triple of circuits.

Assume $\mathcal{C}$ is an orientation of F_7. We show that the WDP of Theorem 3.1(2) cannot be satisfied. Since F_7 is binary, $I = \emptyset$. We may assume that $\mathcal{C}$ contains the signed circuits

$$abc, \quad cfg, \quad \text{and} \quad cde.$$

Since $\big(\{a, b, c\}, \{c, f, g\}, \{a, b, f, g\}\big)$ is a modular triple of circuits, the WDP guarantees that

$$ab\overline{fg}$$

is also an element of $\mathcal{C}$. Similarly, $\mathcal{C}$ contains the signed circuits

$$ab\overline{de} \text{ and } de\overline{fg}.$$

Since $(\{a,e,f\},\{b,d,f\},\{a,b,d,e\})$ is a modular triple of circuits, their signatures in $\mathcal{C}$ must be some reorientation of $ae\overline{f}$, $\overline{bd}f$, and $a\overline{b}de$. But we already know $ab\overline{de}$ is a signed circuit in $\mathcal{C}$, so we must reorient this modular triple by b and e, and possibly by f. Thus $\mathcal{C}$ contains

$$\text{either } \big(a\overline{ef} \text{ and } b\overline{d}f\big) \text{ or } \big(a\overline{e}f \text{ and } b\overline{df}\big). \tag{5.1}$$

Using similar techniques, we find that $\mathcal{C}$ contains

$$\text{either } \big(a\overline{dg} \text{ and } b\overline{e}g\big) \text{ or } \big(a\overline{d}g \text{ and } b\overline{eg}\big). \tag{5.2}$$

But these techniques also show that $\mathcal{C}$ contains

$$\text{either } \big(a\overline{dg} \text{ and } bd\overline{f}\big) \text{ or } \big(ad\overline{g} \text{ and } b\overline{df}\big).$$

The first possibility contradicts (5.1) and the second contradicts (5.2). □

As a final application, we use gains in $\mathbb{Z}_3^+$ to show that the matroid of the graph in Figure 5.2 is an elementary lift of $U_{2,4}$. According to Theorem 4.4, the lifting signature is the ternary signature of $U_{2,4}$, which is $\big\{\overline{12}3, \overline{1}24, 134, \overline{23}4\big\}$. (We found this signature in Section 2.6).

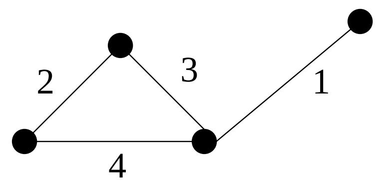


Figure 5.2. The matroid of this graph is a lift of $U_{2,4}$ that can be constructed using gains.

Consider the gain mapping ϕ, where $\phi(2) = \phi(3) = \phi(4) = 1$ and $\phi(1) = 0$. From Section 2.1, we know that

$$L(U_{2,4}, \mathcal{B}(\phi)) = \big((U_{2,4} +_{\mathcal{M}} e)/e\big)^*$$

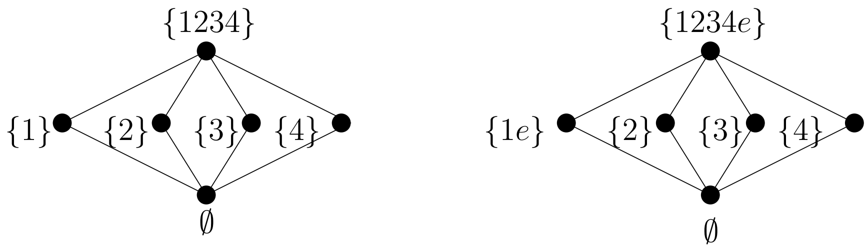


FIGURE 5.3. These are the lattices of $U_{2,4}$ (left) and one if its single-element extensions (right).

is a lift of $U_{2,4}$ where $\mathcal{M}$ is determined by (2.1). Since $\mathcal{B}(\phi) = \{234\}$ and $\mathcal{B}(\phi)^* = \{1\}$, $\mathcal{M} = \{1, 1234\}$. In Figure 5.3, we show the lattices of flats of both $U_{2,4}$ and $U_{2,4} +_{\mathcal{M}} e$. It is easy to see that $(U_{2,4} +_{\mathcal{M}} e)/e$, the dual of $L\big(U_{2,4}, \mathcal{B}(\phi)\big)$, is the matroid of the graph shown in Figure 5.4. It follows that $L\big(U_{2,4}, \mathcal{B}(\phi)\big)$ is the matroid of the graph in Figure 5.2.

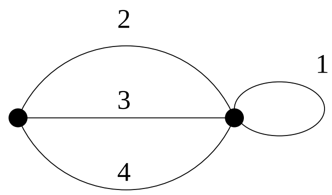


FIGURE 5.4. The matroid $(U_{2,4} +_{\mathcal{M}} e)/e$ is the graphic matroid of this graph.

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