A DESINGULARIZATION OF V_6

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Abstract. We calculate a sequence of blowups to desingularize the projective closure of V_6 - the algebraic curve in the parameter space of quadratic rational maps containing all maps with a period 6 critical point. Our calculations show that the geometric genus of V_6 is 6.

1. Introduction

M_2, the space of holomorphic conjugacy classes of quadratic rational maps is biholomorphic to $\mathbb{C}^2$ (see e.g. [Mi]). V_n is the subset of M_2 consisting of maps having a critical point with period n. The closure of V_n in M_2 is an algebraic curve. In fact, for $n \geq 3$, one may view V_n as the set of parameters $(b, c) \in \mathbb{C}^2$ such that the quadratic map

$$f(z) = 1 + \frac{b}{z} + \frac{c}{z^2}$$

has the n-cycle

$$0 \to \infty \to 1 \to 1 + b + c \to \cdots \to f^n(0) = 0$$

(see e.g. [R]). The equation $f^n(0) = 0$ defines a divisor $D_n = P_n - Q_n$ on $\mathbb{C}^2$, where P_n and Q_n are effective divisors. The closure of V_n in $\mathbb{C}^2$ is (conjecturally) the highest degree prime divisor $\leq P_n$. We will abuse notation slightly and simply call this prime divisor V_n.

For $n = 1, ..., 4$, V_n is a rational curve (i.e., genus 0) while V_5 has geometric genus 1 (see e.g. [GJ]). The intent of this paper is to prove the following result:

Theorem. *The geometric genus of V_6 is 6.*

Our approach to the proof is to resolve the singularities of $\overline{V_6}$ - the projective closure of V_6 in $\mathbb{P}^2(\mathbb{C})$. More specifically, we find blowup sequences to resolve each singular point. A similar approach was used in [GJ] to calculate the genus of V_5. The proofs in [GJ] along with the ones in this paper show that the trees of singularities for V_5 and V_6 are given by figs. 1 and 2, respectively. The apparent connection between the trees for V_5 and for V_6 seems enticing but will not be touched upon here.

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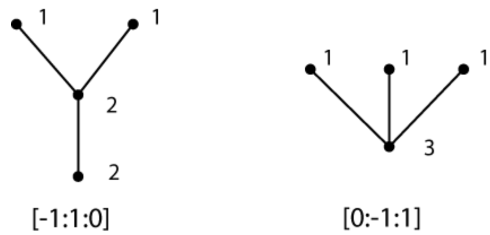

FIGURE 1. V_5**:** The two singularities of $\overline{V_5}$ and their trees of infinitely near points.

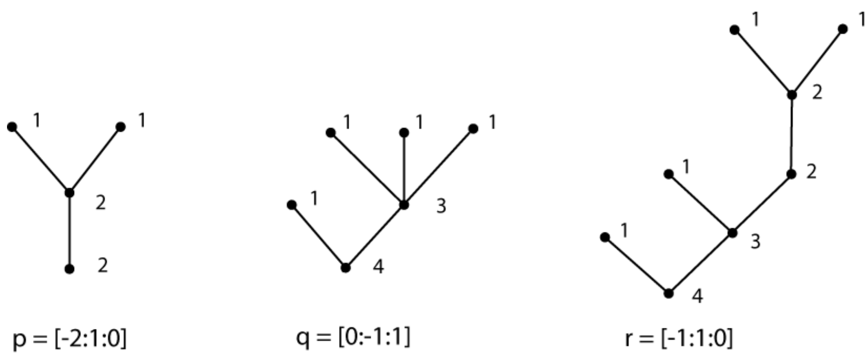

FIGURE 2. V_6**:** The three singularities of $\overline{V_6}$ and their trees of infinitely near points.

This paper is organized as follows: In section 2 we give background information on singular plane curves. In section 3 we show that V_6 is irreducible; and, in section 4 we construct the blowup sequences to resolve V_6.

2. Algebraic Plane Curves

In this section we give the background information on plane curves which we will make use of in subsequent sections. For a more complete treatment on algebraic plane curves the reader may see for example [S] or [C]. An algebraic plane curve C is the zero set of a polynomial p in two variables.

$p = F_0 + F_1 + ... + F_n$, where F_i is homogeneous of degree i. The F_i of lowest degree and non-zero defines the tangent space to C at $(0, 0$.

,V-notation, blowups and infinitely near points, tangents and singularities, changing coordinates in $\mathbb{P}^2$, $C[x, y]$ notation and coefficient matrices and tangents, genus formula. *********

3. Irreducibility of V_6

The expression $f^6(0)$ is given by

$$f^6(0) = \frac{(1+b+c)V}{Q^2},$$

where Q is the defining polynomial for V_5 and V is a degree 9 polynomial. In the projective coordinates $[a:b:c] \in \mathbb{P}^2(\mathbb{C})$, the homogenization of V is:

$$H = a^9+14a^8b+84a^7b^2+283a^6b^3+589a^5b^4+784a^4b^5+666a^3b^6+347a^2b^7+100ab^8+12b^9+14a^8c+161a^7bc+778a^6b^2c+2061a^5b^3c+3271a^4b^4c+3182a^3b^5c+1849a^2b^6c+584ab^7c+76b^8c+77a^7c^2+710a^6bc^2+2688a^5b^2c^2+5419a^4b^3c^2+6288a^3b^4c^2+4197a^2b^5c^2+1487ab^6c^2+214b^7c^2+215a^6c^3+1548a^5bc^3+4456a^4b^2c^3+6586a^3b^3c^3+5275a^2b^4c^3+2166ab^5c^3+354b^6c^3+332a^5c^4+1819a^4bc^4+3863a^3b^2c^4+3979a^2b^3c^4+1985ab^4c^4+382b^5c^4+295a^4c^5+1206a^3bc^5+1809a^2b^2c^5+1179ab^3c^5+281b^4c^5+157a^3c^6+461a^2bc^6+446ab^2c^6+142b^3c^6+51a^2c^7+99abc^7+48b^2c^7+10ac^8+10bc^8+c^9.$$

Using the fact that all singularities of V_n either occur on the line $c=0$ or on the line $a=0$ (see e.g. [St]), it is not hard to check that the singularities of H are at

$$p = [-2:1:0]$$

$$q = [0:1:-1]$$

and

$$r = [1:-1:0].$$

Our first task is to show that H defines $\overline{V_6}$. Since $1+b+c$ defines V_3, we may accomplish this by showing that H is irreducible.

Proposition 1. *H is irreducible and hence $\overline{V_6} = H$.*

Proof. Changing coordinates so that $p=[1:0:0], q=[0:1:0]$, and $r=[0:0:1]$, H has a slightly more manageable form:

$$H_0 = -a^4b^5+a^6b^2c-4a^5b^3c+3a^3b^5c-a^7c^2+8a^6bc^2-24a^5b^2c^2+24a^4b^3c^2-4a^3b^4c^2-3a^2b^5c^2+5a^6c^3-30a^5bc^3+57a^4b^2c^3-42a^3b^3c^3+9a^2b^4c^3+ab^5c^3-7a^5c^4+29a^4bc^4-43a^3b^2c^4+27a^2b^3c^4-6ab^4c^4+2a^4c^5-7a^3bc^5+9a^2b^2c^5-5ab^3c^5+b^4c^5.$$

If $H_0 = H_1H_2$, then by Bezout's theorem, $H_1=0$ intersected with $H_2=0$ will contain $\deg(H_1)\times\deg(H_2)$ points. It is not hard to check that the singularities of $H_0=0$ have multiplicities 2,4, and 4 (in fact, we will find these through the course of the proofs of Lemmas 1-3). Since all intersections of $H_1=0$ with $H_2=0$ will be singular points of $H_0=0$, $\deg(H_1)\times\deg(H_2)\le 10$. Since $\deg(H_0) = \deg(H_1)+\deg(H_2) = 9$, the only possibilities are $1+8$ and $0+9$. We can rule out the case that H_0 has a linear factor since in this case the line would have to pass through two singular points of H_0. In other words, one of a, b, or c would divide H_0. Looking at the above formula for H_0 we see this is clearly not the case. $\square$

4. The Blowup Sequences

The multiplicities of the infinitely near points to p, q and r are given by the following three lemmas. Their proofs produce the trees of singularities illustrated in fig. 2.

Lemma 1. *The infinitely near points to* $p = [-2:1:0]$ *have multiplicities 2,2,1 and 1.*

Proof. First we change coordinates to move p to $[1:0:0]$. In this case the tangent space is given by $(b+c)^2 = 0$. We make a further change of coordinates to move $b + c = 0$ to $b = 0$. (The overall coordinate change may be accomplished by $a \mapsto a, b \mapsto -a/2 + 2b - c$, and $c \mapsto -b + c$.) We denote the polynomial defining V_6 on the affine patch $\mathbb{A}^2_a = \{a = 1\}$ by $P(b, c)$. The polynomial $P \in \mathbb{C}[b, c]$ has coefficient matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 8 & -10 & -8 \\ 1 & -14 & 60 & -92 & 104 & 0 \\ 9 & -96 & 304 & -208 & -192 & 32 \\ -16 & 210 & -784 & 1408 & -160 & -32 \\ -342 & 2328 & -3312 & -320 & 416 & 0 \\ -616 & 1632 & 3712 & -2112 & 0 & 0 \\ 2224 & -8800 & 5376 & 0 & 0 & 0 \\ 7008 & -6944 & 0 & 0 & 0 & 0 \\ 3680 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this form it is easy to see that the tangent space of V_6 at $(b, c) = (0, 0)$ is the double tangent $b^2 = 0$. Thus we will blowup the origin in $\mathbb{A}^2_a$.

We write the blowup $\pi : B \to \mathbb{A}^2_a$ explicitly as

$$B = \{(b, c; s : t) \mid at = cs\} \subset \mathbb{A}^2_a \times \mathbb{P}^1,$$

where $[s : t]$ are the projective coordinates on $\mathbb{P}^1$. The pullback of V_6 on $\mathbb{A}^2_a \times \mathbb{A}^1_s$ is $P(ct, c) = c^2 P_2$, where P_2 is the polynomial in $\mathbb{C}[c, t]$ with coefficient matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -14 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 60 & -96 & -16 & 0 & 0 & 0 & 0 & 0 \\ 2 & -10 & -92 & 304 & 210 & -342 & 0 & 0 & 0 & 0 \\ 0 & -8 & 104 & -208 & -784 & 2328 & -616 & 0 & 0 & 0 \\ 0 & 0 & 0 & -192 & 1408 & -3312 & 1632 & 2224 & 0 & 0 \\ 0 & 0 & 0 & 32 & -160 & -320 & 3712 & -8800 & 7008 & 0 \\ 0 & 0 & 0 & 0 & -32 & 416 & -2112 & 5376 & -6944 & 3680 \end{pmatrix}$$

Since $c = 0$ is the local equation for the exceptional divisor, the zero set of P_2 is the birational transform of $H = 0$. From the coefficient matrix for P_2 we see that the tangent space at the double point $(0,0;1:0)$ consists of the two distinct tangent lines $\{t(t-b)=0\}\cap\{c=0\}$. Thus the singularity will resolve into two smooth points after one more blowup. $\square$

In the proofs of the following lemmas we will be somewhat more loose with our notation for blowups than we were for Lemma 1. In particular, we will only keep track of the birational transforms of V_6 and not the blowup equations defining the twisted planes. No information is lost with this approach, but the authors find that the presentation is improved considerably.

Lemma 2. *The infinitely near points to $q=[0:-1:1]$ have multiplicities 4,3,1,1,1,and 1.*

Proof. We first change coordinates to move q to $[0:1:0]$ and the tangent $a+c=0$ to $b=0$. (This may be accomplished by $a\mapsto a-c, b\mapsto b$, and $c\mapsto -b+c$.) Then on $\mathbb{A}^2_b$, V_6 is the zero set of the polynomial $Q\in\mathbb{C}[a,c]$ whose coefficient matrix is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1\\
0 & 0 & 0 & 1 & -4 & 1 & 11 & -14 & 5 & 0\\
0 & 0 & 0 & 5 & -18 & 15 & 4 & -6 & 0 & 0\\
0 & 0 & 3 & 4 & -32 & 33 & -8 & 0 & 0 & 0\\
0 & 0 & 10 & -5 & -26 & 19 & 0 & 0 & 0 & 0\\
0 & 1 & 18 & -23 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 3 & 16 & -16 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Since the tangent space is $ac^3=0$, blowing up will result in a smooth point and a triple point. Let us examine the local equation at the triple point:

$$Q(a,at)=-a^4Q_2,$$

where $Q_2\in\mathbb{C}[a,t]$ has coefficient matrix

$$\begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -3 & -5 & 4 & 1 & 0 & 0 & 0 & 0\\
0 & -1 & -10 & -4 & 18 & -1 & -4 & 0 & 0 & 0\\
0 & -3 & -18 & 5 & 32 & -15 & -11 & 6 & 0 & 0\\
0 & -7 & -16 & 23 & 26 & -33 & -4 & 14 & -4 & 0\\
-1 & -5 & -1 & 16 & -1 & -19 & 8 & 6 & -5 & 1
\end{pmatrix}$$

At $(a,t)=(0,0)$, the birational transform has 3 distinct tangents given by $-a^2t-3at^2-t^3=-t(a+\frac{3+\sqrt{5}}{2}t)(a+\frac{3-\sqrt{5}}{2}t)=0$, and hence one more blowup will resolve the triple point into 3 smooth points. $\square$

Lemma 3. *The infinitely near points to $r = [-1:1:0]$ have multiplicities 4,3,2,2,1,1,1, and 1.*

Proof. First we change coordinates to move r to [1:0:0] and the tangent $2b+c=0$ to $c=0$. (This may be accomplished by $a \mapsto a, b \mapsto -a+b$, and $c \mapsto -2b+c$.) Now, on $\mathbb{A}^2_a$, V_6 is the zero set of the polynomial $R \in \mathbb{C}[b,c]$ whose coefficient matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -5 & 3 & -8 & 0 \\ 0 & 0 & 0 & -2 & 6 & 18 & -22 & 32 & 0 & 0 \\ 0 & 1 & -1 & -22 & -21 & 67 & -82 & 0 & 0 & 0 \\ 0 & 4 & 26 & 3 & -115 & 145 & 0 & 0 & 0 & 0 \\ -1 & -16 & 15 & 122 & -180 & 0 & 0 & 0 & 0 & 0 \\ 5 & -15 & -79 & 154 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 28 & -86 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix we see that the tangents are $b^3c = 0$. Thus blowing up will result in a smooth point and a triple point. We examine the local equation for the birational transform of V_6 near the triple point:

$$R(cs, c) = -c^4 R_2,$$

where $R_2 \in \mathbb{C}[c,s]$ has coefficient matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -6 & 22 & -26 & 16 & -5 & 0 & 0 & 0 \\ 0 & 5 & -18 & 21 & -3 & -15 & 15 & -5 & 0 & 0 \\ 0 & -3 & 22 & -67 & 115 & -122 & 79 & -28 & 4 & 0 \\ -1 & 8 & -32 & 82 & -145 & 180 & -154 & 86 & -28 & 4 \end{pmatrix}$$

Thus the tangents are $-c^2s + 2cs^2 - s^3 = -(c-s)^2 s = 0$, and so the next blowup will result in a smooth point and a double point. First we change coordinates to move $c - s = 0$ to $c = 0$ ($c \mapsto c - s$ and $s \mapsto -s$). Then the local equation for the birational transform at the double point is

$$R_2(sx, s) = -s^3 R_3,$$

where $R_3 \in \mathbb{C}[s,x]$ has coefficient matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 & 0 \\ -4 & 31 & -60 & 40 & -8 & 1 \\ -26 & 140 & -249 & 189 & -62 & 8 \\ -86 & 405 & -715 & 591 & -227 & 32 \\ -177 & 815 & -1460 & 1265 & -525 & 82 \\ -252 & 1168 & -2137 & 1923 & -847 & 145 \\ -254 & 1201 & -2259 & 2111 & -979 & 180 \\ -182 & 882 & -1708 & 1652 & -798 & 154 \\ -90 & 446 & -884 & 876 & -434 & 86 \\ -28 & 140 & -280 & 280 & -140 & 28 \\ -4 & 20 & -40 & 40 & -20 & 4 \end{pmatrix}$$

Now the tangents are given by the double line $-4s^2 + 4sx - x^2 = -(2s-x)^2 = 0$. After a coordinate change to move $2s - x = 0$ to $x = 0$ ($s \mapsto -s$ and $x \mapsto -2s - s$), blowing up one last time results in a birational transform whose local equation is $R_4 = 0$ where $R_4 \in \mathbb{C}[s, v]$ has coefficient matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 5 & 9 & 0 & 0 & 0 \\ -6 & -40 & -30 & 5 & 0 & 0 \\ 43 & 111 & 9 & -40 & 0 & 0 \\ -98 & -33 & 227 & 125 & -8 & 0 \\ -10 & -516 & -678 & -135 & 52 & -1 \\ 416 & 1209 & 645 & -231 & -147 & 8 \\ -646 & -682 & 761 & 997 & 205 & -32 \\ -8 & -1598 & -2810 & -1385 & -27 & 82 \\ 1068 & 3300 & 2868 & 380 & -471 & -145 \\ -1080 & -1724 & 224 & 1692 & 1002 & 180 \\ -64 & -1632 & -3544 & -2968 & -1106 & -154 \\ 832 & 2880 & 3760 & 2360 & 720 & 86 \\ -576 & -1600 & -1760 & -960 & -260 & -28 \\ 128 & 320 & 320 & 160 & 40 & 4 \end{pmatrix}$$

In this case the double point has two distinct tangents

$$0 = -6s^2 + 5sv - v^2 = -(2s - v)(3s - v),$$

and so it will be resolved into 2 smooth points after another blowup. □

For the interested reader, a Mathematica notebook containing the calculations described in the proofs of Lemmas 1-3 may be found on the third author's website (see URL below).

Since Lemmas 1-3 tell us the multiplicities of all the infinitely near points to the singularities of $\overline{V_6}$, we may apply the genus formula to obtain:

Theorem. *The geometric genus of V_6 is 6.*

Collecting results up to this point, we have:

Corollary. *The geometric genus of* V_n *is* 0 *for* $1 \leq n \leq 4$*, 1 for* $n = 5$*, and* 6 *for* $n = 6$*.*

5. References

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