ON QUADRATIC RATIONAL MAPS WITH A PERIODIC CRITICAL POINT

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Abstract. We generate detailed computer images of the hyperbolic components of the parameter spaces $V_1 - V_4$, distinguishing between each type of component. We also calculate a desingularization for the projective closure of V_5, showing that as an algebraic curve, the geometric genus of V_5 is 1.

1. Introduction

A rational map $f : \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \to \mathbb{P}^1$ is hyperbolic if under iteration, each critical point of f is attracted to some attracting periodic cycle. Hyperbolic maps are an open and (conjecturally) dense set in the space of rational maps (see e.g. [MSS]). Connected components of the set of hyperbolic maps are called hyperbolic components.

V_n is the set of holomorphic conjugacy classes of quadratic rational maps with a critical point of period n. For example, V_1 may be identified with the family of quadratic polynomials

$$z \mapsto z^2 + c, \ c \in \mathbb{C},$$

each map having ∞ as a fixed critical point. The dynamics of the maps in V_1 are encoded by the much studied Mandelbrot set (see e.g. [BM],[M],[DH]).

The V_n's have been studied for the past several decades and much progress has been made (especially for $n \leq 3$) in describing the hyperbolic components of these parameter spaces (see e.g. [R1-3],[W],[T]). For small values of n, V_n identifies with $\mathbb{C}$, and so one may study its dynamical plane.

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In the case of $n \geq 3$, one can take V_n to be the set of $(b, c) \in \mathbb{C}^2$ such that for the quadratic map

$$f(z) = 1 + b/z + c/z^2,$$

the critical point 0 has period n (see e.g. [R3]). So, for $n \geq 3$, any map f in V_n has the critical cycle

$$0 \mapsto \infty \mapsto 1 \mapsto ... \mapsto f^{n-2}(1) = 0.$$

For $n \geq 3$, $f^{n-2}(1) = 0$ may be written in the form

$$\frac{P_n}{Q_n} = 0,$$

where P_n and Q_n are polynomials in b and c having no common factors. The closure of V_n (in $\mathbb{C}^2$) is a complex algebraic curve supported on $P_n = 0$. For example, since $f(1) = 1 + b + c$, V_3 identifies with the complex line

$$P_3 = 1 + b + c = 0.$$

For $n = 4$, we have

$$f^2(1) = \frac{1 + 3b + 2b^2 + 3c + 3bc + c^2}{P_3^2}.$$

Thus V_4 is contained in the zero set of the irreducible conic

$$P_4 = 1 + 3b + 2b^2 + 3c + 3bc + c^2.$$

In this paper we will only consider the cases $n \leq 5$. (See section 4 for P_5.)

For $n = 3$ and 4, the projective closure of V_n, which we will denote by $\overline{V_n}$, is birational to $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. However, as we prove in section 5, the geometric genus of $\overline{V_5}$ is 1, i.e. $\overline{V_5}$ is birational to a torus. This means that in this higher period case, V_n is not injectively parametrized by $\mathbb{C}$ and so the dynamical plane method does not apply.

This paper is organized as follows: In section 2 we recall the classification of hyperbolic quadratic rational maps with a critical cycle. We then explain how this classification gives an algorithm for approximating the hyperbolic components of V_n, while distinguishing between their types. In section 3 we illustrate some computer generated representations of the hyperbolic components of V_n for $n = 1, ..., 4$. Section 4 contains our calculation of the genus of V_5. Section 5 contains some closing remarks about V_n; while, section 6 describes the computer program we wrote to produce the graphics in this paper.

2. Hyperbolic Quadratic Rational Maps

Since quadratic maps have 2 critical points and any map f in V_n has the critical cycle

$$0 \mapsto \infty \mapsto 1 \mapsto ... \mapsto f^{n-2}(1) = 0,$$

we shall refer to the one remaining critical point of f as the free critical point. Hyberbolic rational maps have been classified by their critical orbits (see e.g. [R1-3]). In fact, any hyperbolic map f in V_n must be exactly one of the following types:

Type 1 The free critical point is attracted to the fixed critical point.

Type 2 The free critical point is in a periodic component of the attracting basin of the critical cycle.

Type 3 The free critical point is in a preperiodic component of the attracting basin of the critical cycle.

Type 4 The free critical point belongs to the attracting basin of a periodic orbit other than the critical cycle.

This classification suggests an algorithm for making an approximation of the hyperbolic components of V_n. Note that it is easy to distinguish type 4 mappings from the other types by testing the orbit of the free critical point for attraction to any points in the critical cycle. Also, if $n > 1$ then V_n has no type 1 maps, and if $n = 1$ then V_n has no maps of types 2 or 3. So the case of $n = 1$ is easy; and, for $n > 1$ we just need to distinguish between type 2 and type 3 maps. To do this, one must decide if there is a path from the free critical point to the attracting periodic point, entirely contained within the immediate attracting basin. For many type 2 maps in V_n, the line segment between the free critical point and its attractor lies within the immediate basin. However, there are type 2 maps in V_3 and V_4 for which a nonlinear path must be found (see Figs. 1,7, and 9). We solved this problem by flood-filling the immediate attracting basin to test for the free critical point.

3. Computer Generated Images

Using the above algorithm (and a parametrization for V_n), we can generate a graphical approximation of the hyperbolic components of V_n - distinguishing between types. In this section we illustrate such approximations for $n =$

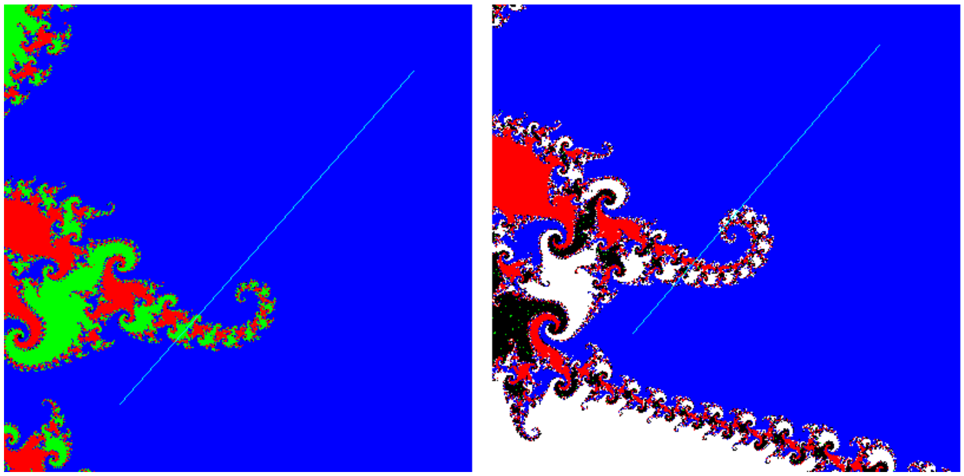

FIGURE 1. The image on the left shows the basins of attraction for the map $f(z) = (z-c)(z-1)/z^2$ where $c = .16-2.2i$. The free critical point is attracted to 1 under iteration of f^3 in this case, but the line segment joining the free critical point and 1 is not entirely contained in the immediate basin. The image on the right shows a similar situation for a map in V_4.

$1, 2, 3$, and 4. Although approximations of the hyperbolic components of V_1, V_2, and V_3 have previously been illustrated (see e.g. [B-M],[M], and [W]), we will include these parameter spaces for the reader's convenience. Next we will provide some representations of V_4.

V_1**:** The set of holomorphic conjugacy classes of quadratic rational maps with a fixed critical point may be identified with the family of polynomials

$$f(z) = z^2 + c \text{ where } c \in \mathbb{C}.$$

There are no type II or III components in this case. The complement of the single type I component is the classical Mandelbrot set (see Fig. 2).

V_2**:** The set of holomorphic conjugacy classes of quadratic rational maps with a period 2 critical point may be identified with the rational map

$$f(z) = \frac{1}{z^2}$$

together with the family

$$f(z) = \frac{a}{z^2 + 2z}, \ a \neq 0.$$

V_2 contains one type 2 component (see Fig. 3). A detailed study of V_2 has been given in [T].

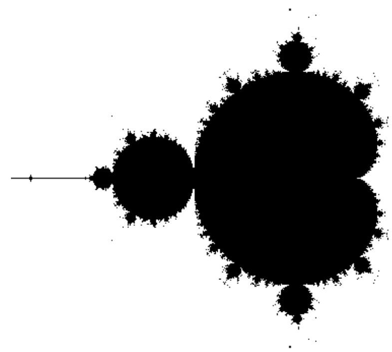

FIGURE 2. V_1: The single type 1 component is in white. Since ∞ is a fixed critical point there are no maps of types 2 or 3. The free critical point, for the maps colored in black, is not attracted to ∞.

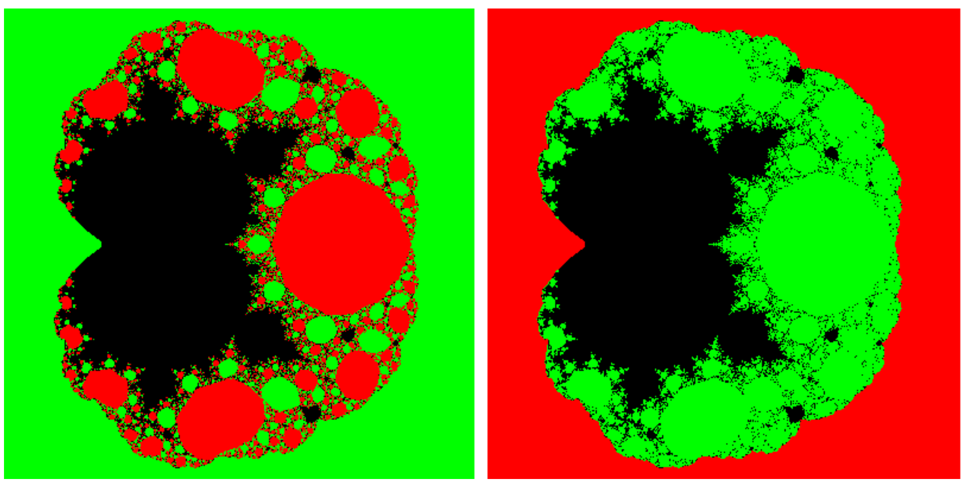

FIGURE 3. V_2: The image on the left shows the maps whose free critical point is attracted to ∞ in red, while the maps with attraction to 0 are colored green. The image on the right side shows the type 2 hyperbolic component in red and the type 3 components in green. The free critical point, for the maps colored in black, is not attracted to the critical cycle.

$V_n, n \geq 3$: In this case, V_n is the collection of quadratic functions

$$f(z) = 1 + \frac{b}{z} + \frac{c}{z^2}$$

that have the critical n-cycle

$$0 \mapsto \infty \mapsto 1 \mapsto ... \mapsto f^{n-2}(1) = 0.$$

For example, V_3 is defined by

$$f(1) = 1 + b + c = 0,$$

which gives us the one parameter family of maps

$$f(z) = 1 + \frac{-1-c}{z} + \frac{c}{z^2} = \frac{(z-1)(z-c)}{z^2}.$$

V_3 has 2 type II components: one containing $c = -1$ and the other containing $c = 1$ (see Fig. 4). A nearly complete topological description of the hyperbolic components of V_3 has been given in [R3].

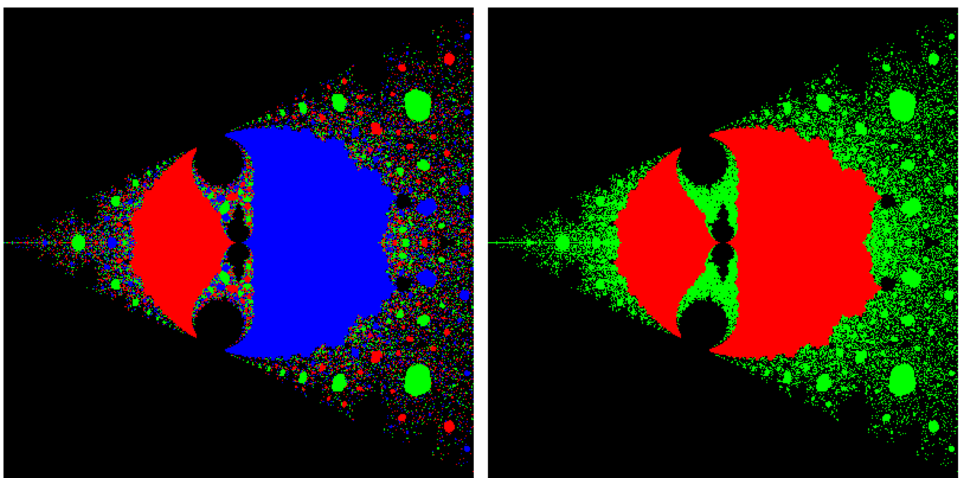

FIGURE 4. V_3: The image on the left shows the maps whose free critical point is attracted to ∞, 0, and 1 in red, green, and blue, respectively. The image on the right shows the type 2 maps in red and the type 3 maps in green. The free critical point, for the maps colored in black, is not attracted to the critical cycle.

V_4 is determined by $f^2(1) = 0$, which defines the algebraic curve

$$1 + 3b + 2b^2 + 3c + 3bc + c^2 = 0.$$

One may find a rational parametrization for an irreducible conic by projecting from any point on the curve (see e.g. [S]). Figures 5 and 6 show some representations of V_4 using two different parametrizations. Our images show V_4 to have 6 type 2 components, which is confirmed in [KR].

Both V_3 and V_4 required the flood-fill algorithm to distinguish their type 2 components from their type 3 components (see Fig. 7). V_2 does not seem to have any such maps.

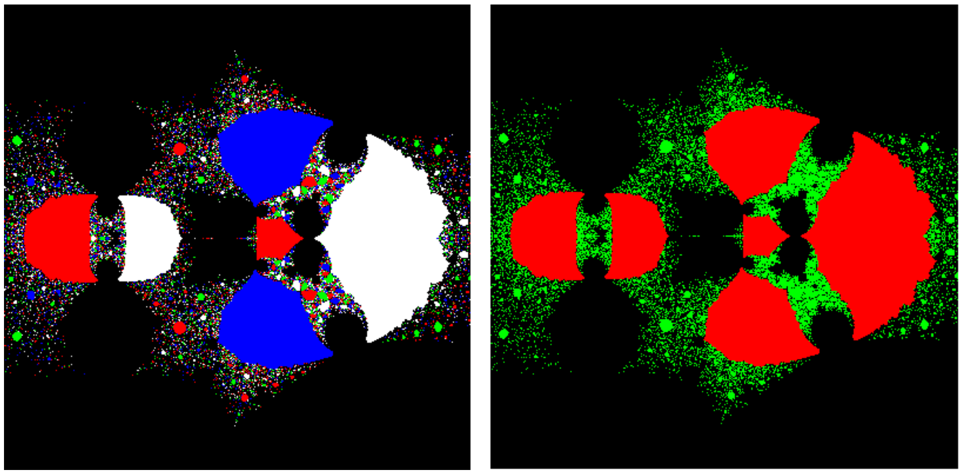

FIGURE 5. V_4 projected from $(-1, 0)$: The image on the left shows the maps whose free critical point is attracted to ∞, 0, 1, and $1+b+c$ in red, green, blue, and white, respectively. The image on the right shows the type 2 components in red, while the green represents the type 3 components. The free critical point, for the maps colored in black, is not attracted to the critical cycle.

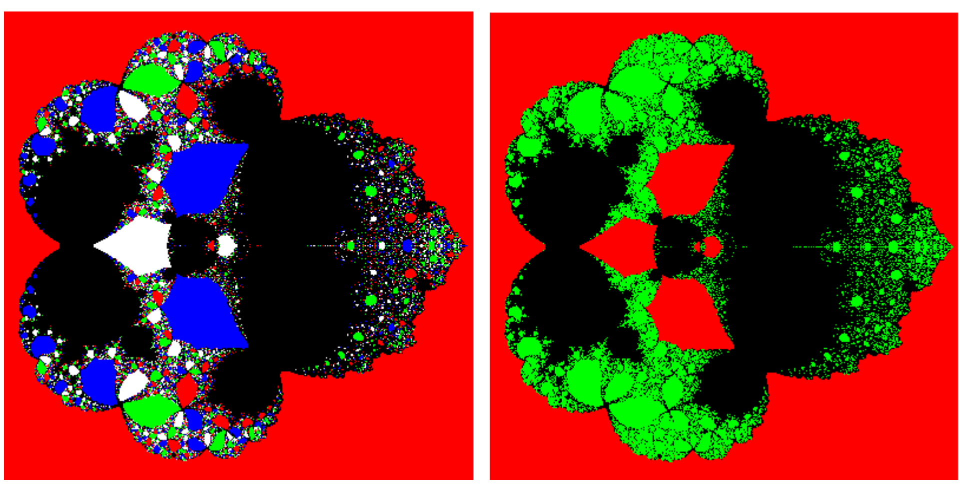

FIGURE 6. V_4 projected from $(0, (3+\sqrt{5})/2)$: The coloring is the same as in Fig. 5.

4. THE GENUS OF V_5

It is a classical result that any singular point p on an algebraic curve C may be resolved by a sequence of blowups (see e.g. [S] or [C]). The point p and

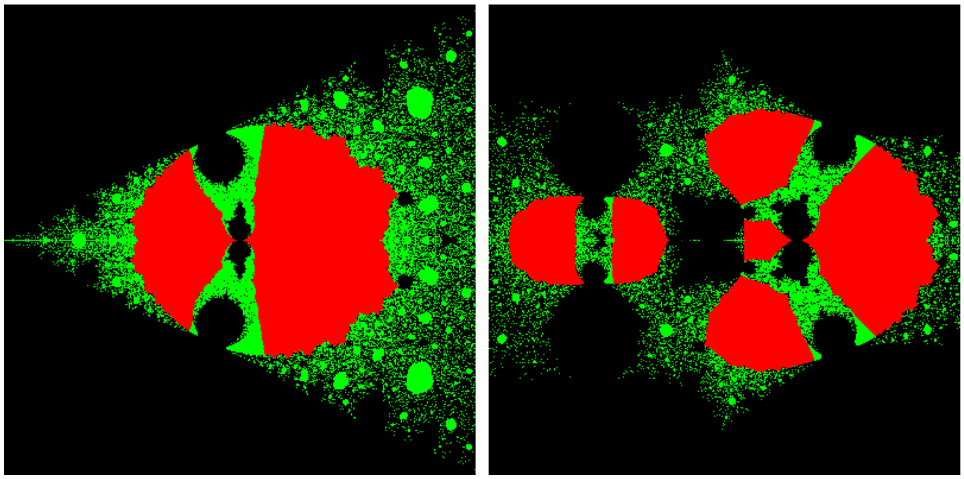

FIGURE 7. The red approximates those type 2 maps for which the line segment between the free critical point and its attracting periodic point is entirely contained within the immediate attracting basin. V_3 is on the left, while V_4 is on the right. Compare with Figs. 4 and 5.

the singular points arising from these blowups are called the infinitely near points to p.

To calculate the geometric genus of an irreducible complex projective plane algebraic curve $C \subset \mathbb{P}^2$ one may use the genus formula:

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum \frac{k_i(k_i-1)}{2},$$

where d is the degree of C and the k_i's are the multiplicities of all the infinitely near points to the singularities of C (see e.g. [S]).

Geometrically, blowing-up a point consists of replacing the point by a line of tangent directions. Algebraically, the blowup of a point in affine space $\mathbb{A}^n = \{(a_1, a_2, ..., a_n) | a_i \in \mathbb{C}\}$ is described as follows:

After suitable change of coordinates one may arrange that the point to blowup is the origin $(0, 0, ..., 0) \in \mathbb{A}^n$. In this case, setting

$$B = \{(a_1, a_2, ..., a_n; x_1 : x_2 : ... : x_n) | a_i x_j = a_j x_j \text{ for all } i, j\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1},$$

the birational map $\pi : B \to \mathbb{A}^n$ given by

$$\pi(a_1, a_2, ..., a_n; x_1 : x_2 : ... : x_n) = (a_1, a_2, ..., a_n),$$

is called the blowup of $\mathbb{A}^n$ centered at the origin.

For more information on plane curves, singularities, and blowups the reader may refer to a resource such as [S] or [C].

For $n \geq 3$, V_n is contained in a complex plane algebraic curve defined by

$$f^{n-2}(1) = 0,$$

where

$$f(z) = 1 + \frac{b}{z} + \frac{c}{z^2}.$$

In what follows, by the genus of V_n we will mean the geometric genus of the projective closure of V_n in $\mathbb{P}^2$.

V_5 is defined by

$$f^3(1) = 0,$$

which simplifies to

$$\frac{P_5}{P_4^2},$$

where P_4 is the polynomial defining V_4 (see above) and P_5 is a degree 5 polynomial in b and c. The homogenization of P_5 is

$$\begin{aligned} H(a,b,c) = {} & a^5 + 7a^4b + 18a^3b^2 + 21a^2b^3 + 11ab^4 + 2b^5 + \\ & 7a^4c + 33a^3bc + 53a^2b^2c + 35ab^3c + 8b^4c + \\ & 15a^3c^2 + 44a^2bc^2 + 42ab^2c^2 + 13b^3c^2 + \\ & 12a^2c^3 + 23abc^3 + 11b^2c^3 + \\ & 5ac^4 + 5bc^4 + c^5, \end{aligned}$$

(i.e. $P_5 = H(1, b, c)$.) In what follows we shall refer to the complex projective plane algebraic curve $H = 0$ simply as H. In the projective coordinates $[a : b : c]$, the singularities of H are at

$$p = [0 : 1 : -1]$$

and

$$q = [-1 : 1 : 0].$$

The infinitely near points to p and q are described by the following 2 lemmas.

Lemma: The infinitely near points to p have multiplicities 3,1,1, and 1.

Proof: After changing coordinates, we may assume that p occurs at the origin $(a, c) = (0, 0)$ on the affine patch $\mathbb{A}_b^2 = \{b = 1\} \subset \mathbb{P}^2$. In this case, the local equation for H is

$$a^2c+3ac^2+c^3 = -(a^5+3a^3c+7a^4c+8a^2c^2+15a^3c^2+3ac^3+12a^2c^3+5ac^4+c^5).$$

Since $a^2c+3ac^2+c^3 = c(a+\frac{3+\sqrt{5}}{2}c)(a+\frac{3-\sqrt{5}}{2}c)$, H has three distinct tangents at p. $\square$

Lemma: The infinitely near points to q have multiplicities 2,2, and 1.

Proof: We change coordinates so that q occurs at the origin $(a, c) = (0, 0)$ in $\mathbb{A}_b^2$. Then the local equation for H is

$$a^2 = g(a, c), \text{ where}$$

$$g(a, c) = 2a^4 + a^5 - 4a^2c + 5a^3c + 7a^4c- \\ ac^2 - a^2c^2 + 15a^3c^2 - ac^3 + 12a^2c^3 + 5ac^4 + c^5.$$

Hence q is a cusp of multiplicity 2. We will resolve q by blowing up.

Let $\pi : B \to \mathbb{A}_b^2$ be the blowup of $\mathbb{A}_b^2$ at the origin. We can write B explicitly as

$$B = \{(a, c; x : y)|ay = cx\} \subset \mathbb{A}_b^2 \times \mathbb{P}^1,$$

where $[x : y]$ are the projective coordinates on $\mathbb{P}^1$. Then H is birationally equivalent to the projective closure of

$$V = \pi_1^{-1}(\{a^2 = g(a, c)\} - \{(0, 0)\}) \cap (\mathbb{A}_b^2 \times \mathbb{P}^1)$$
$$= \{(a, c; x : y)|a^2 = g(a, c), ay = cx, (a, c) \neq (0, 0)\}.$$

If we set $\infty = (0, 0; 1 : 0)$, then on the affine patch $\mathbb{A}_b^2 \times \mathbb{A}_y^1$

$$V = \{(a, c, x)|a^2 = g(a, c), a = cx, c \neq 0\} \cup \infty$$

$$= \{(a, c, x)|(cx)^2 = g(cx, c), a = cx, c \neq 0\} \cup \infty$$

$$= \{(a, c, x)|c^2x^2 = c^2(c^3 - cx - c^2x + 5c^3x - 4cx^2 - c^2x^2 + 12c^3x^2+$$

$$5c^2x^3 + 15c^3x^3 + 2c^2x^4 + 7c^3x^4 + c^3x^5), a = cx, c \neq 0\} \cup \infty$$

$$= \{(a, c, x)|x(x + c) = c^3 - c^2x + 5c^3x - 4cx^2 - c^2x^2 + 12c^3x^2+$$

$$5c^2x^3 + 15c^3x^3 + 2c^2x^4 + 7c^3x^4 + c^3x^5, a = cx, c \neq 0\} \cup \infty.$$

This curve has distinct tangent lines $\{a = 0, x = 0\}$ and $\{a = 0, x + c = 0\}$ at $(0, 0, 0)$; therefore, q will be resolved after one more blowup.□

Proposition. H is irreducible, and hence $\overline{V_5} = H$.

Proof: If $H = C_1 \cup C_2$, then $C_1 \cap C_2$ will be singular points of H. By Bezout's theorem, $\#(C_1 \cap C_2) = \deg C_1 \times \deg C_2$. Counting multiplicities, H has 5 singularities and so $\deg C_1 \times \deg C_2 \leq 5$. Thus $\deg C_1 + \deg C_2 = \deg H = 5$ implies that the degrees of C_1 and C_2 must be 1 and 4. Let us suppose C_1 is a line and C_2 is a quartic. Since $\#(C_1 \cap C_2) = 4$ and H has two singularities, C_1 must be tangent to C_2 at one of the singularities of H. From the proof of our first lemma, the singularity p has 3 distinct tangents and hence C_1 is not tangent to C_2 at p. Thus C_1 must be tangent to C_2 at the double point q. Now, Bezout's theorem implies C_1 must intersect C_2 two more times. Since C_1 is not tangent at p, it must meet C_2 at a point other than p or q. This is contrary to H having only two distinct singular points. □

Applying the genus formula, we have:

Theorem. The geometric genus of V_5 is 1.

5. Conclusions

Formulas are calculated in [KR] for the number of type 2 (and type 4) hyperbolic components in V_n for any n. Their results verify that the type 2 components we illustrated in section 3 (see Figs. 3 - 6) make up the complete set of type 2 components in V_n for $n \leq 4$.

In general, the structure of V_n is linked to the structure of V_1. For example, the number of type 4 components in V_n (for a fixed period of the attracting cycle) depends on the number of hyperbolic components in certain limbs of V_1 (see [KR]). A more simplistic observation is that, in some sense, each V_n appears to contain infinitely many topologically equivalent copies of V_1 (see Fig. 7). Indeed, the Mandelbrot set seems to be a common locus for maps with bounded critical points in parameter spaces of quadratic (and quadratic type) maps (see e.g. [DJS]). For more information on the dynamical and topological structure of V_n (and other quadratic maps) the reader may see, for example, [Mi], [R1-3], and [T].

A complex projective curve C may be parametrized by rational functions if and only if its desingularization is isomorphic to $\mathbb{P}^1$. This is equivalent to the geometric genus of C being 0 (see e.g. [S]). For $n = 1, ..., 4$ the genus of V_n is 0, and so one may use the dynamical plane of V_n to describe the family of maps in these cases. However, as we showed, the genus of V_5 is 1 and so other methods must be found to illustrate this parameter space.

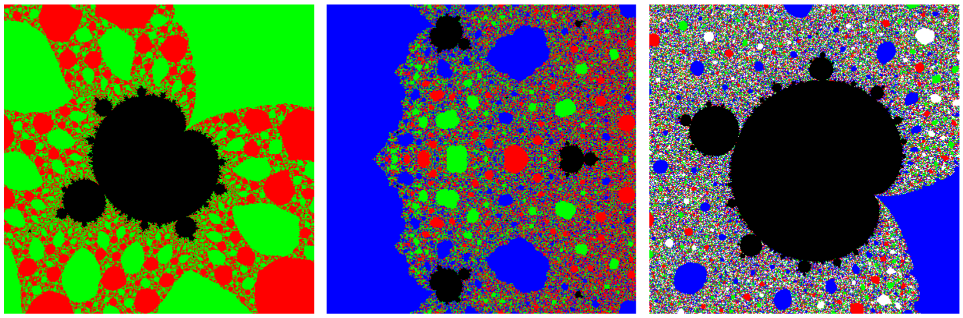

FIGURE 8. From left to right are zoomed-in sections of V_2, V_3, and V_4. Each V_n appears to contain infinitely many copies of V_1.

Calculating the genus of V_n is certainly not limited to what we have done above. In theory, one can desingularize any curve by a sequence of blow-ups. However, as n gets larger, the singularity structure of $\overline{V_n}$ becomes more complex, making a desingularization for $\overline{V_n}$ more difficult to calculate. During preparation of this paper, the second author and two undergraduate students calculated a blow-up sequence to desingularize $\overline{V_6}$. That calculation, which may be found in [DHJ], shows the genus of V_6 to be 6. More on the singularities of $\overline{V_n}$ may be found in [RS].

6. Generating the Computer Images

The images in this paper were generated by a Java applet written by the authors (see Fig. 9). This zoomable fractal generator may be used to graphically explore V_1, V_2, V_3, and V_4 (and other spaces of quadratic rational maps) in a much more detailed manner than given in this paper. The applet and source code are freely available for use and/or download at either authors' websites (see URLs below). Full screenshots of the images in this paper (including input data) are available on the second author's website.

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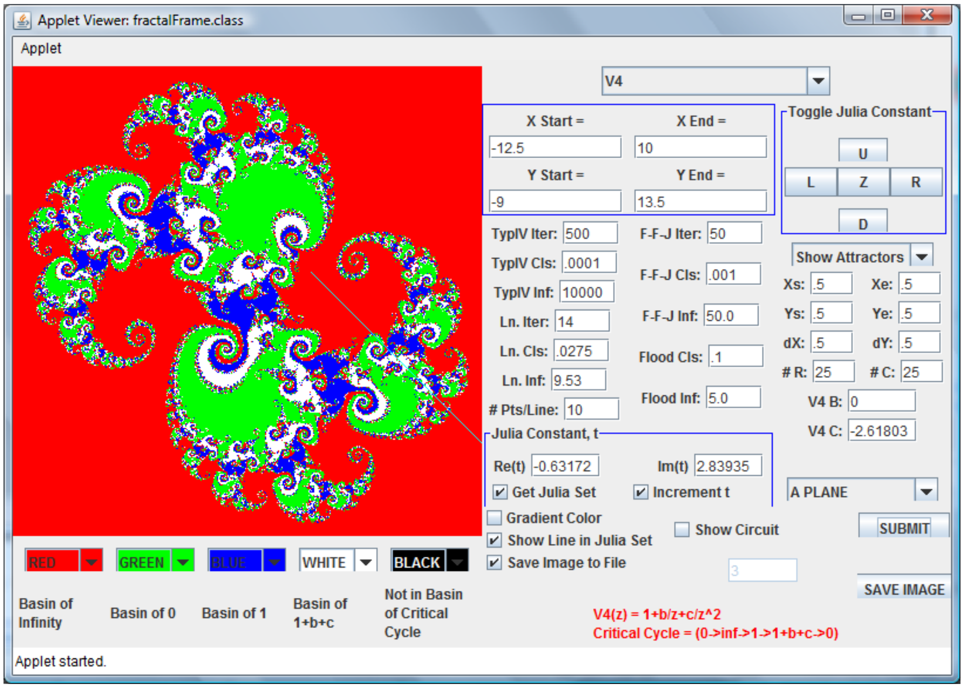

FIGURE 9. Screenshot: The basins of attraction for a type 2 map f in V_4. The free critical point is attracted to ∞ under iteration of f^4. The line segment fails to connect the free critical point with ∞.

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