PROPERTIES OF BOURBAKI'S FUNCTION

JAMES MCCOLLUM

Abstract. We examine Bourbaki's Function, an easily-constructed continuous but nowhere-differentiable function, and explore properties including functional identities, the antiderivative, and the Hausdorff dimension of the graph.

1. Introduction

While Bernard Bolzano [1] introduced one of the earliest examples of a continuous, nowhere-differentiable function, his example is only one of countless similar functions, many of which are defined in less complex ways. One such function is found in Nicolas Bourbaki's *Elements of Mathematics—Functions of a Real Variable* [2]: This function, which we call Bourbaki's Function, is defined by a few inductive rules, and its simple self-similar structure allows for abundant and relatively easy analysis.

Okamoto [3] defines Bourbaki's Function f_i for any iteration $i \geq 0$ over $[0, 1]$ as follows: $f_0(x) = x$ for all $x \in [0, 1]$, every f_i is continuous on $[0, 1]$, every f_i is affine in each subinterval $[k/3^i, (k+1)/3^i]$ where $k \in \{0, 1, 2, \dots, 3^i - 1\}$, and

$$f_{i+1}\left(\frac{k}{3^i}\right) = f_i\left(\frac{k}{3^i}\right), \tag{1}$$

$$f_{i+1}\left(\frac{3k+1}{3^{i+1}}\right) = f_i\left(\frac{k}{3^i}\right) + \frac{2}{3}\left[f_i\left(\frac{k+1}{3^i}\right) - f_i\left(\frac{k}{3^i}\right)\right], \tag{2}$$

$$f_{i+1}\left(\frac{3k+2}{3^{i+1}}\right) = f_i\left(\frac{k}{3^i}\right) + \frac{1}{3}\left[f_i\left(\frac{k+1}{3^i}\right) - f_i\left(\frac{k}{3^i}\right)\right], \tag{3}$$

$$f_{i+1}\left(\frac{k+1}{3^i}\right) = f_i\left(\frac{k+1}{3^i}\right). \tag{4}$$

Figs. 1 and 2 illustrate the construction of the graph of f. Okamoto [3] has shown, using the above equations, that the function

$$f(x) = \lim_{i\to\infty} f_i(x)$$

is continuous and nowhere differentiable. We can observe from these equations that $f_i(x) = f(x)$ for any $x \in [0, 1]$ that can be expressed as some multiple of $1/3^i$. The values between each of these inputs, however, may change with new iterations.

But for f, formulas (1)–(4) can evaluate more than just multiples of powers of one third. Consider the ternary expansion $x = 0.x_1x_2 \dots x_i$, where $x_1, x_2, \dots, x_i \in \{0, 1, 2\}$. (If $x = 1$, we say equivalently that $x = 0.222\dots$) The value of i coincides with the first iteration i for which $f_i(x) = f(x)$. We can rewrite the ternary expansion as

$$x = \sum_{j=1}^{i} \frac{x_j}{3^j}.$$

Date: August 2010.

1991 *Mathematics Subject Classification.* 26A30,37C45.

Key words and phrases. continuous, nowhere differentiable, fractal, Hausdorff dimension.

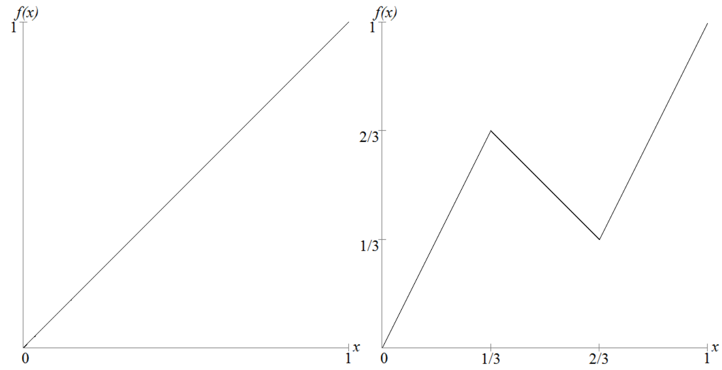

FIGURE 1. Graphs of f_0 and f_1. Note the steps taken in constructing f_1 from f_0.

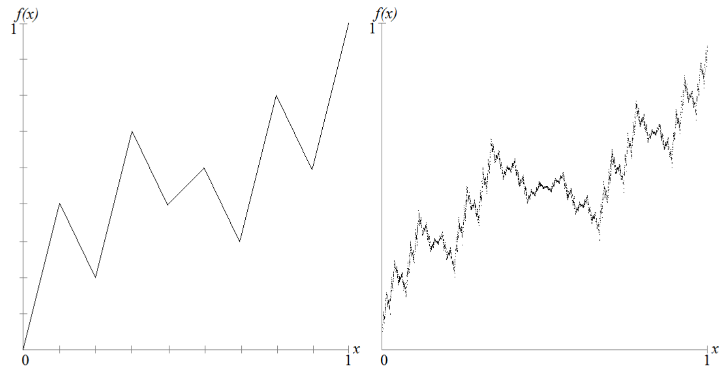

FIGURE 2. Graph of f_2 and an approximate graph of f. Again, note the steps taken in constructing f_2 from f_1.

Since the sum always will have a denominator of $1/3^i$, we can evaluate $f(x)$ using an f_i that covers intervals of length at least $1/3^i$—which, by definition, is f_h for $h \geq i$. To evaluate $f(x)$, we start with $k = 0$ and $i = 1$, and we apply formula (1), (2), or (3) depending on the value of each x_j: If $x_j = 0$, we use (1); if $x_j = 1$, we use (2); and if $x_j = 2$, we use (3). In every case, we must keep track of the values $k/3^i$, $(3k+1)/3^{i+1}$, $(3k+2)/3^{i+1}$, $(3k+1)/3^i$, and $(3k+2)/3^i$.

Of course, for every f_i where i is finite, we will have $x \in \{0, 1/3^i, 2/3^i, \ldots, 1\}$. To evaluate $f(x)$ for some x that cannot be expressed in this form—an irrational number or any rational number in $[0,1]$ whose denominator is not a power of three—we must consider a non-terminating ternary expansion:

$$x = \sum_{j=1}^{\infty} \frac{x_j}{3^j}.$$

In other words, to evaluate $f(x)$ for such an x, we would have to apply formulas (1), (2), and (3) indefinitely. While we certainly could obtain a fair approximation with enough iterations, keeping track of certain values—in particular, $f_i((k+1)/3^i) - f_i(k/3^i)$—would grow more difficult with each iteration. Nevertheless, we see that formulas (1), (2), and (3) will provide values of $f(x)$ for all $x \in [0,1]$, including the values that do not directly take the form of an integer over a power of three.

In this paper, we will use the self-similarity of f to find "shortcuts" to evaluating $f(x)$ for such values of x, and we will examine the relationship of this self-similarity to the fractal nature of the graph of f, to the antiderivative F of f, and to the properties of the graph of F. In Section 2, we will prove that the graph of f possesses rotational symmetry about the point $(1/2, 1/2)$ and equivalently that $f(1-x) = 1 - f(x)$ for all $x \in [0,1]$. In the same section, we will use the function's self-similar properties to infer three basic identities that evaluate $f(x/3^i)$, $f([2-x]/3^i)$, and $f([2+x]/3^i)$ in terms of $f(x)$. In Section 3, we will use these general identites to evaluate $f(x)$ for specific sets of numbers that have some form other than $x = k/3^i$. In Section 4, we will use the rotational symmetry of the graph of f to prove that $\int_x^{1-x} f(t)\,dt = 1/2 - x$. In the same section, we will derive three other identities for the area under the graph of f, and we will use these identities iteratively to construct a graph of F. Using the Fundamental Theorem of Calculus, we will show that F is a continuous function that is neither concave up nor concave down anywhere. In Section 5, we will do for F what we did for f in Section 3. Finally, in Section 6, we will show that the graph of f has Hausdorff dimension $\log_3 5$, while the graph of F has Hausdorff dimension 1.

2. Functional Identities

We observe from the graphs that each f_i possesses rotational symmetry about its center, which implies a useful identity:

Theorem 1. *For all* $x \in [0,1]$, $f(1-x) = 1 - f(x)$.

Proof. By definition, $f_0(x) = x$ for all $x \in [0,1]$, so clearly this is true for f_0. To prove this for f itself, however, we will take advantage of the function's inductive nature and consider only the points where x can be expressed as a rational number in $[0,1]$ whose denominator takes the form 3^i.

We consider $i = 1$ for a base case. We must show, then, that $f_1(1-x) = 1 - f(x)$ for $x \in \{0, 1/3, 2/3, 1\}$. Evaluating the function at these points, we get

$$\begin{aligned} f_1(1-0) &= f_1(1) = 1 = 1 - 0 = 1 - f_1(0) \\ f_1\left(1 - \frac{1}{3}\right) &= f_1\left(\frac{2}{3}\right) = \frac{1}{3} = 1 - \frac{2}{3} = 1 - f_1\left(\frac{1}{3}\right) \\ f_1\left(1 - \frac{2}{3}\right) &= f_1\left(\frac{1}{3}\right) = \frac{2}{3} = 1 - \frac{1}{3} = 1 - f_1\left(\frac{2}{3}\right) \\ f_1(1-1) &= f_1(0) = 0 = 1 - 1 = 1 - f_1(1) \end{aligned}$$

So for $i = 1$, $f_i(1-x) = 1 - f_i(x)$ for all $x \in \{0, 1/3^i, 2/3^i, \ldots, 1\}$.

From here we can make our inductive hypothesis: For some $j \geq 1$, $f_j(1-x) = 1 - f_j(x)$, where $x \in \{0, 1/3^j, 2/3^j, \ldots, 1\}$. Now, we can deduce from (1) and (4) that if $f_j(1-x) = 1 - f_j(x)$ and $f_{j+1}(x) = f_j(x)$ for $x = k/3^j$, then $f_{j+1}(1-x) = 1 - f_{j+1}(x)$, since both $x, (1-x) \in \{0, 1/3^j, 2/3^j, \ldots, 1\}$. To apply (2) and (3), we first let $k' = 3^j - 1 - k$. This means that $k' \in \{0, 1, \ldots, 3^j - 1\}$ and that $1 - k/3^j = (k'+1)/3^j$. Then

$$\begin{aligned} f_{j+1}\left(1 - \frac{3k+1}{3^{j+1}}\right) &= f_{j+1}\left(\frac{3^{j+1} - 3k - 1}{3^{j+1}}\right) \\ &= f_{j+1}\left(\frac{(3^{j+1} - 3k - 3) + 2}{3^{j+1}}\right) \\ &= f_{j+1}\left(\frac{3k' + 2}{3^j}\right) \\ &= f_j\left(\frac{k'}{3^j}\right) + \frac{1}{3}\left[f_j\left(\frac{k'+1}{3^j}\right) - f_j\left(\frac{k'}{3^j}\right)\right]. \end{aligned}$$

Using our inductive hypothesis, we make appropriate substitutions for these iteration-j functions to get

$$\begin{aligned} f_{j+1}\left(1-\frac{3k+1}{3^{j+1}}\right) &= 1-f_j\left(1-\frac{k'}{3^j}\right)+\frac{1}{3}\left[1-f_j\left(1-\frac{k'+1}{3^j}\right)\right] \\ &\quad -\frac{1}{3}\left[1-f_j\left(1-\frac{k'}{3^j}\right)\right] \\ &= 1-f_j\left(\frac{k+1}{3^j}\right)+\frac{1}{3}f_j\left(\frac{k+1}{3^j}\right)-\frac{1}{3}f_j\left(\frac{k}{3^j}\right) \\ &= 1-\frac{2}{3}f_j\left(\frac{k+1}{3^j}\right)+\frac{2}{3}f_j\left(\frac{k}{3^j}\right)-f_j\left(\frac{k}{3^j}\right) \\ &= 1-\left(f_j\left(\frac{k}{3^j}\right)+\frac{2}{3}\left[f_j\left(\frac{k+1}{3^j}\right)-f_j\left(\frac{k}{3^j}\right)\right]\right) \\ &= 1-f_{j+1}\left(\frac{3k+1}{3^{j+1}}\right). \end{aligned}$$

Working the same substitutions through for $f_{j+1}(1-(3k+2)/3^{j+1})$ will give us $1-f_{j+1}((3k+2)/3^{j+1})$.

Therefore, by induction, $f_i(1-x) = 1-f_i(x)$ for all $i \geq 1$ and all $x \in \{0, 1/3^i, 2/3^i, \ldots, 1\}$. As i approaches infinity, the interval $1/3^i$ between each $x \in \{0, 1/3^i, 2/3^i, \ldots, 1\}$ approaches zero, and since this set is dense in $[0,1]$, the limit $f(x)$ satisfies $f(1-x) = 1-f(x)$ for all $x \in [0,1]$. □

This identity proves helpful in evaluating $f(x)$ for values of x whose ternary expansions do not terminate. We only have to find these values for either the left half or the right half of $[0,1]$, and we will have the values for the other half readily. Given the function's rotational symmetry and its domain, it should be clear that $f(1/2) = 1/2$. With the identity, we can show this concisely:

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 1-f\left(\frac{1}{2}\right) \\ 2f\left(\frac{1}{2}\right) &= 1 \\ f\left(\frac{1}{2}\right) &= \frac{1}{2} \end{aligned}$$

Katsuura [4] defines the contraction mappings $w_n : X \mapsto X$, where $n \in \{1,2,3\}$ and $X = [0,1]\times[0,1]$, as follows: For all $(x,y) \in X$,

$$w_1(x,y) = \left(\frac{x}{3}, \frac{2y}{3}\right) \tag{5}$$

$$w_2(x,y) = \left(\frac{2-x}{3}, \frac{1+y}{3}\right) \tag{6}$$

$$w_3(x,y) = \left(\frac{2+x}{3}, \frac{1+2y}{3}\right) \tag{7}$$

Seperately applying mappings (5), (6), and (7) to the line $y = x$ (the graph of f_0) produces the graph of f_1; applying the same mappings to the graph of f_1 gives the graph of f_2; and so on. More generally, if Γ_i is the graph of f_i, then

$$\Gamma_{i+1} = w_1(\Gamma_i) \cup w_2(\Gamma_i) \cup w_3(\Gamma_i).$$

Since $f = \lim_{i\to\infty} f_i$, we can say that $\Gamma = \lim_{i\to\infty} \Gamma_{i+1} = \lim_{i\to\infty} w_1(\Gamma_i) \cup w_2(\Gamma_i) \cup w_3(\Gamma_i)$. So $\Gamma = w_1(\Gamma) \cup w_2(\Gamma) \cup w_3(\Gamma)$ is the unique invariant set for the iterated function system (IFS) given by w_1, w_2, and w_3 (See [4]). Since $w_1(\Gamma_i) = \Gamma_{i+1}$ on $[0,1/3]$,

$w_2(\Gamma_i) = \Gamma_{i+1}$ on $[1/3, 2/3]$, and $w_3(\Gamma_i) = \Gamma_{i+1}$ on $[2/3, 1]$, we are able to prove three more identities:

Proposition 1. *For all* $x \in [0,1]$ *and* $i \geq 0$, $f\left(\left(\frac{1}{3}\right)^i x\right) = \left(\frac{2}{3}\right)^i f(x)$.

Proof. Let $x \in [0,1]$. Then $(x, f(x)) \in \Gamma$. If $i = 0$, our result is obvious. If $i > 0$, then

$$\underbrace{w_1 \circ w_1 \circ \cdots \circ w_1}_{i-1} \circ w_1(x, f(x)) = \left(\left(\frac{1}{3}\right)^i x, \left(\frac{2}{3}\right)^i f(x)\right)$$

from the definition of w_1. And since $w_1^i(\Gamma) \subseteq \Gamma$, where $w_1^i(\Gamma)$ denotes i applications of w_1 on Γ, we have $f\left(\left(\frac{1}{3}\right)^i x\right) = \left(\frac{2}{3}\right)^i f(x)$. □

Proposition 2. *For all* $x \in [0,1]$ *and* $i > 0$,

$$f\left(\frac{2-x}{3^i}\right) = \frac{2^{i-1}}{3^i}[1 + f(x)].$$

Proof. Let $x \in [0,1]$. Then $(x, f(x)) \in \Gamma$. If $i > 0$, then

$$\begin{aligned}\underbrace{w_1 \circ w_1 \circ \cdots \circ w_1}_{i-1} \circ w_2(x, y) &= \left(\frac{([2-x]/3)}{3^{i-1}}, \frac{2^{i-1}([1+y]/3)}{3^{i-1}}\right) \\ &= \left(\frac{2-x}{3^i}, \frac{2^{i-1}(1+y)}{3^i}\right)\end{aligned}$$

from the definition of w_1. And since $w_1(\Gamma) \subseteq \Gamma$ and $w_2^i(\Gamma) \subseteq \Gamma$, we have $f\left(\frac{2-x}{3^i}\right) = \frac{2^{i-1}}{3^i}[1 + f(x)]$. □

Proposition 3. *For all* $x \in [0,1]$ *and* $i > 0$,

$$f\left(\frac{2+x}{3^i}\right) = \left(\frac{2}{3}\right)^i f(x) + \frac{2^{i-1}}{3^i}.$$

Proof. This can be proven in the same manner as Proposition 2 if we replace the w_2 with w_3. □

3. Function Values

As f has no explicit formula, we must take advantage of its self-similar structure to evaluate $f(x)$ for nearly all values of x (see Fig. 3). The four identities we have proven already will help.

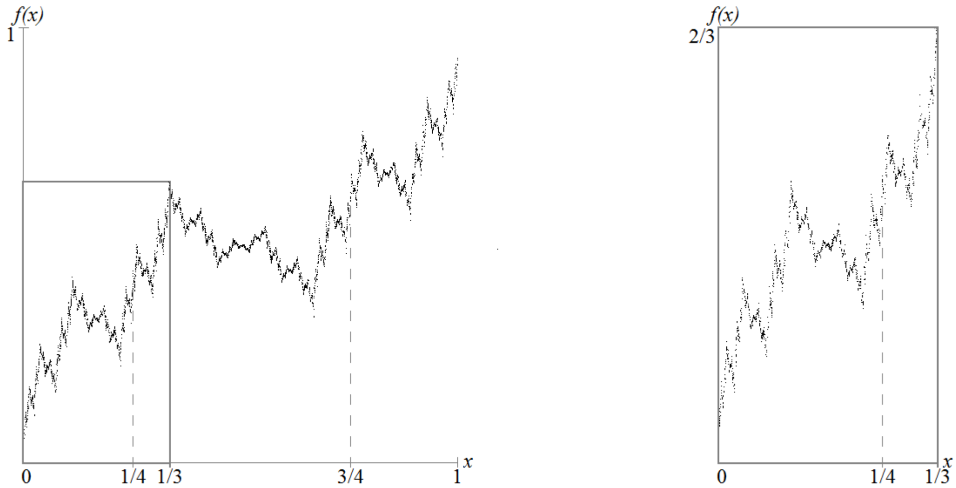

FIGURE 3. As f has no explicit formula, the self-similarity of its graph is key in determining its values at different points.

Theorem 2. *For all* $j > i > 0$,

(i) $f\left(\frac{1}{3^i+1}\right) = \frac{2^i}{3^i+2^i}$,

(ii) $f\left(\frac{1}{3^i-1}\right) = \frac{2^i}{3^i+2^{i-1}}$,

(iii) $f\left(\frac{2}{3^i+1}\right) = \frac{2^{i-1}}{3^i-2^{i-1}}$,

(iv) $f\left(\frac{2}{3^i-1}\right) = \frac{2^{i-1}}{3^i-2^i}$,

(v) $f\left(\frac{1}{3^j+3^i}\right) = \left(\frac{2}{3}\right)^i \left(\frac{2^{j-i}}{3^{j-i}+2^{j-i}}\right)$, *and*

(vi) $f\left(\frac{1}{3^j-3^i}\right) = \left(\frac{2}{3}\right)^i \left(\frac{2^{j-i}}{3^{j-i}+2^{j-i-1}}\right)$.

Proof. Let $j > i > 0$.

(i) Clearly

$$1 - \frac{1}{3^i+1} = \frac{3^i}{3^i+1}$$

and using Theorem 1 and Proposition 1, it follows that

$$\begin{aligned} f\left(\frac{1}{3^i+1}\right) &= f\left(\frac{1}{3^i}\left[\frac{3^i}{3^i+1}\right]\right) \\ &= f\left(\frac{1}{3^i}\left[1-\frac{1}{3^i+1}\right]\right) \\ &= \left(\frac{2}{3}\right)^i f\left(1-\frac{1}{3^i+1}\right) \\ &= \left(\frac{2}{3}\right)^i \left[1-f\left(\frac{1}{3^i+1}\right)\right] \\ &= \left(\frac{2}{3}\right)^i - \left(\frac{2}{3}\right)^i f\left(\frac{1}{3^i+1}\right). \end{aligned}$$

So we have

$$\left[1+\left(\frac{2}{3}\right)^i\right] f\left(\frac{1}{3^i+1}\right) = \left(\frac{2}{3}\right)^i$$

and thus,

$$f\left(\frac{1}{3^i+1}\right) = \frac{2^i}{3^i+2^i}, \text{ for } i>0.$$

(ii) Likewise, we know that for $i>0$,

$$\frac{1}{3^i-1} = 1-\frac{3^i-2}{3^i-1}$$

and

$$\frac{2-(3^i-2)/(3^i-1)}{3^i} = \frac{1}{3^i-1}.$$

The next steps are almost identical to those from the previous proof, so we will omit them. Making appropriate substitutions, applying the function to both sides, and using Theorem 1 and Proposition 2 gives us

$$f\left(\frac{1}{3^i-1}\right) = \frac{2^i}{3^i+2^{i-1}}, \text{ for } i>0.$$

(iii) For $i>0$, we have

$$\frac{2-(2/(3^i+1))}{3^i} = \frac{2}{3^i+1}.$$

Applying the function to both sides and using Proposition 2 gives us

$$f\left(\frac{2}{3^i+1}\right) = \frac{2^{i-1}}{3^i-2^{i-1}}, \text{ for } i>0.$$

(iv) Similarly,

$$\frac{2+(2/(3^i-1))}{3^i} = \frac{2}{3^i-1},$$

and by applying f to both sides and using Proposition 3, we get

$$f\left(\frac{2}{3^i-1}\right) = \frac{2^{i-1}}{3^i-2^i}, \text{ for } i>0.$$

(v) Now, for $j>i>0$, we know that

$$\frac{1}{3^j+3^i} = \left(\frac{1}{3^i}\right)\left(\frac{1}{3^{j-i}+1}\right)$$

and since $j > i > 0$ implies that $j - i > 0$, we can apply Proposition 1 and the identity $f(1/(3^i+1)) = 2^i/(3^i+2^i)$, which we proved in part (i) of this theorem to obtain

$$\begin{aligned} f\left(\frac{1}{3^j+3^i}\right) &= f\left(\left(\frac{1}{3^i}\right)\left(\frac{1}{3^{j-i}+1}\right)\right) \\ &= \left(\frac{2}{3}\right)^i f\left(\frac{1}{3^{j-i}+1}\right) \\ &= \left(\frac{2}{3}\right)^i \left(\frac{2^{j-i}}{3^{j-i}+2^{j-i}}\right), \text{ for } j > i > 0. \end{aligned}$$

(vi) Given $j > i > 0$, we also know that

$$\frac{1}{3^j-3^i} = \left(\frac{1}{3^i}\right)\left(\frac{1}{3^{j-i}-1}\right)$$

and by applying Proposition 1 and the identity $f(1/(3^i-1)) = 2^i/(3^i+2^{i-1})$, which we proved in part (ii) of this theorem, we get

$$\begin{aligned} f\left(\frac{1}{3^j-3^i}\right) &= f\left(\left(\frac{1}{3^i}\right)\left(\frac{1}{3^{j-i}-1}\right)\right) \\ &= \left(\frac{2}{3}\right)^i f\left(\frac{1}{3^{j-i}-1}\right) \\ &= \left(\frac{2}{3}\right)^i \left(\frac{2^{j-i}}{3^{j-i}+2^{j-i-1}}\right) \text{ for } j > i > 0. \quad \square \end{aligned}$$

4. Integral Identities

What can we say about the integral of f? Is it a continuous but nowhere differentiable function, as f is, or does it exhibit less pathological behavior? What fractal properties does it exhibit, if it exhibits any? What would the graph of f's antiderivative look like?

We might make a starting guess that the integral, as it measures the area under a self-similar curve, will exhibit a degree of self-similarity itself. It turns out that this is the case: We can derive four identities for the integral from our identities for f.

Theorem 3. *For all* $x \in [0,1]$, $\displaystyle\int_x^{1-x} f(t)\,dt = 1/2 - x$.

Proof. By Theorem 1, we know that for any $t \in [0,1]$, $f(1-t) = 1 - f(t)$. So clearly

$$\begin{aligned} \int_a^b f(1-t)\,dt &= \int_a^b 1 - f(t)\,dt \\ &= [b-a] - \int_a^b f(t)\,dt \end{aligned}$$

for all $a, b \in [0,1]$. So if we let $a = x$ and $b = 1 - x$, where $x \in [0,1]$, we get

$$\begin{aligned} \int_x^{1-x} f(1-t)\,dt &= [(1-x)-x] - \int_x^{1-x} f(t)\,dt \\ &= [1-2x] - \int_x^{1-x} f(t)\,dt. \end{aligned}$$

By u-substitution on $\int_x^{1-x} f(1-t)\,dt$, we have

$$\begin{aligned}\int_x^{1-x} f(1-t)\,dt &= -\int_{1-(x)}^{1-(1-x)} f(t)\,dt \\ &= -\int_{1-x}^{x} f(t)\,dt \\ &= \int_x^{1-x} f(t)\,dt.\end{aligned}$$

Then by substitution,

$$\int_x^{1-x} f(t)\,dt = [1-2x] - \int_x^{1-x} f(t)\,dt.$$

So

$$2\int_x^{1-x} f(t)\,dt = 1-2x$$

and thus,

$$\int_x^{1-x} f(t)\,dt = \frac{1}{2} - x, \text{ for all } x \in [0,1]. \qquad \square$$

This theorem is illustrated in Fig. 4. One notable result immediately follows:

Corollary. *The area under the graph of Bourbaki's Function is*

$$A = \int_0^1 f(t)\,dt = \frac{1}{2}.$$

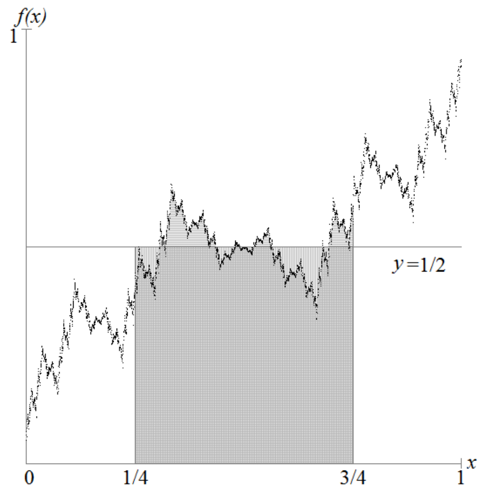

FIGURE 4. The symmetry of the curve generated by f applies to the area under it, as well; over any region $[x, 1-x]$ for $x \in [0,1]$, the area under the curve is equal to the area under the line $y = 1/2$.

Proposition 4. *For all $x \in [0,1]$ and $i \geq 0$,* $\displaystyle\int_0^{x/3^i} f(t)\,dt = \left(\frac{2}{9}\right)^i \int_0^x f(t)\,dt.$

Proof. By Proposition 1, for all $t, x \in [0,1]$ and $i \geq 0$,

$$\int_0^x f\left(\frac{t}{3^i}\right)\,dt = \int_0^x \left(\frac{2}{3}\right)^i f(t)\,dt = \left(\frac{2}{3}\right)^i \int_0^x f(t)\,dt.$$

Now

$$\int_0^x f\left(\frac{t}{3^i}\right)\,dt = 3^i \int_0^{x/3^i} f(t)\,dt = \left(\frac{2}{3}\right)^i \int_0^x f(t)\,dt$$

and thus,

$$\int_0^{x/3^i} f(t)\,dt = \left(\frac{2}{9}\right)^i \int_0^x f(t)\,dt.$$ □

This result is illustrated in Fig. 5.

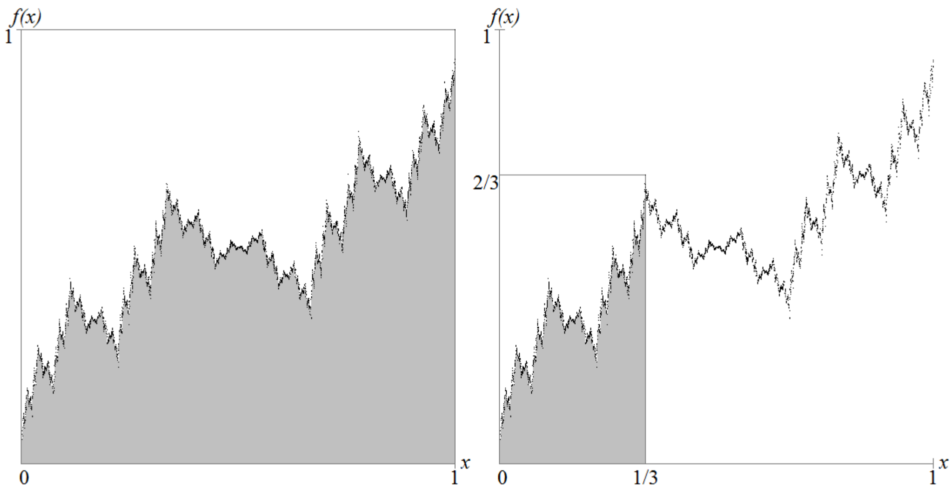

FIGURE 5. Proposition 4 for $i = 1$. The area under the graph over $[0,1]$ and the area over $[0,1/3]$ are in the proportion 1:2/9. This is exactly the proportion of the area in the two boxes pictured.

Proposition 5. *For all $x \in [0,1]$ and $i > 0$,*

$$\int_{(2-x)/3^i}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[x + \int_0^x f(t)\,dt\right].$$

Proof. We know by Proposition 2 that for all $t \in [0,1]$ and $i > 0$, $f([2-t]/3^i) = (2^{i-1}/3^i)[1 + f(t)]$. So clearly

$$\begin{aligned}\int_0^x f\left(\frac{2-t}{3^i}\right)dt &= \int_0^x \frac{2^{i-1}}{3^i}[1+f(t)]\,dt \\ &= \frac{2^{i-1}}{3^i}\int_0^x 1 + f(t)\,dt \\ &= \frac{2^{i-1}}{3^i}\left[x + \int_0^x f(t)\,dt\right]\end{aligned}$$

for all $x \in [0,1]$. By u-substitution on $\int_0^x f([2-t]/3^i)\,dt$, we have

$$\begin{aligned}\int_0^x f\left(\frac{2-t}{3^i}\right)dt &= -3^i \int_{2/3^i}^{(2-x)/3^i} f(t)\,dt \\ &= 3^i \int_{(2-x)/3^i}^{2/3^i} f(t)\,dt.\end{aligned}$$

So

$$3^i \int_{(2-x)/3^i}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{3^i}\left[x + \int_0^x f(t)\,dt\right]$$

and thus,

$$\int_{(2-x)/3^i}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[x + \int_0^x f(t)\,dt\right], \text{ for all } x \in [0,1] \text{ and } i \geq 0. \qquad \square$$

Proposition 6. *For all $x \in [0,1]$ and $i > 0$,*

$$\int_{(2+x)/3^i}^{2/3^i} f(t)\,dt = \left(\frac{2^{i-1}}{9^i}\right)x + \left(\frac{2}{9}\right)^i \int_0^x f(t)\,dt.$$

Proof. This can be proven in the same manner as Proposition 5 if we apply Proposition 3 instead of Proposition 2. $\square$

Using Propositions 4–6, we can construct a simple inductive formula for the antiderivative of f.

Theorem 4. *The antiderivative F of f can be expressed as $F(x) = \lim_{i\to\infty} F_i(x)$, where F_i is defined at any iteration $i \geq 0$ as follows: $F_0(x) = x/2$ for all $x \in [0,1]$, every F_i is continuous on $[0,1]$, every F_i is affine on each subinterval $[k/3^i, (k+1)/3^i]$ where $k \in \{0,1,2,\ldots,3^i-1\}$, and*

$$F_{i+1}\left(\frac{k}{3^i}\right) = F_i\left(\frac{k}{3^i}\right), \tag{8}$$

$$F_{i+1}\left(\frac{k/3^i}{3}\right) = \frac{2}{9}F_i\left(\frac{k}{3^i}\right), \tag{9}$$

$$F_{i+1}\left(\frac{1+k/3^i}{3}\right) = \frac{1}{9}\left[1 + \frac{2k}{3^i} - F_i\left(\frac{k}{3^i}\right)\right], \tag{10}$$

$$F_{i+1}\left(\frac{2+k/3^i}{3}\right) = \frac{1}{9}\left(\frac{5}{2} + \frac{k}{3^i}\right) + \frac{2}{9}F_i\left(\frac{k}{3^i}\right) \tag{11}$$

$$F_{i+1}\left(\frac{k+1}{3^i}\right) = F_i\left(\frac{k+1}{3^i}\right) \tag{12}$$

Proof. Given the domain of f, we will let the antiderivative $F(x) = \displaystyle\int_0^x f(t)\,dt$. Using this notation, Proposition 4 can be expressed as $F\left(\frac{x}{3^i}\right) = \left(\frac{2}{9}\right)^i F(x)$. We also can rewrite Propositions 5 and 6 accordingly, but we must adjust them so that their integrals have a lower bound of 0. We can do this easily for Proposition 5 using a few substitutions:

$$\begin{aligned}
F\left(\frac{1+x}{3^i}\right) &= \int_0^{1/3^i} f(t)\,dt + \int_{1/3^i}^{(1+x)/3^i} f(t)\,dt \\
&= \int_0^{1/3^i} f(t)\,dt + \int_{1/3^i}^{(2-[1-x])/3^i} f(t)\,dt \\
&= \int_0^{1/3^i} f(t)\,dt + \int_{1/3^i}^{2/3^i} f(t)\,dt - \int_{(2-[1-x])/3^i}^{2/3^i} f(t)\,dt \\
&= \left(\frac{2}{9}\right)^i\left(\frac{1}{2}\right) + \frac{2^{i-1}}{9^i}\left(\frac{3}{2}\right) - \frac{2^{i-1}}{9^i}[(1-x) + F(1-x)] \\
&= \frac{2^{i-1}}{9^i}\left[\frac{3}{2} + x - F(x) - \frac{1}{2} + x\right] \\
&= \frac{2^{i-1}}{9^i}[1 + 2x - F(x)].
\end{aligned}$$

Working out Proposition 6 is even simpler:

$$\begin{aligned}
F\left(\frac{2+x}{3^i}\right) &= \int_0^{(2+x)/3^i} f(t)\,dt \\
&= \int_0^{1/3^i} f(t)\,dt + \int_{1/3^i}^{2/3^i} f(t)\,dt + \int_{2/3^i}^{(2+x)/3^i} f(t)\,dt \\
&= \left(\frac{2}{9}\right)^i\left(\frac{1}{2}\right) + \frac{2^{i-1}}{9^i}\left(\frac{3}{2}\right) + \frac{2^{i-1}}{9^i}x + \left(\frac{2}{9}\right)^i F(x) \\
&= \frac{2^{i-1}}{9^i}\left(\frac{5}{2} + x\right) + \left(\frac{2}{9}\right)^i F(x).
\end{aligned}$$

For $i = 1$, the expressions in Propositions 4–6 describe the area under the graph of f over each third of $[0, 1]$ in terms of the area over $[0, 1]$. For $i = 2$, Propositions 4–6 can be applied over one another to describe the area under the graph over each third of each third of $[0, 1]$ in terms of the areas for $i = 1$, and so on. Using our three rewritten propositions, we can approximate the graph of F with continuous, affine iterations. We start with the area over $[0, 1]$: We know that $F(0) = 0$, and from the corollary to Theorem 3, we have $F(1) = 1/2$, so our first iteration must be the graph of $F_0(x) = x/2$. By applying Propositions 4–6 from here, we can evaluate $F_i(k/3^i)$ for any $k \in \{0, 1, 2, \ldots, 3^i - 1\}$. Finally, because the set of all $k/3^i$ is dense in $[0, 1]$ as i goes to infinity, we have $F(x) = \lim_{i\to\infty} F_i(x)$ for all $x \in [0, 1]$. □

Using this theorem, we can obtain a decent approximation of the graph of F (see Fig. 6).

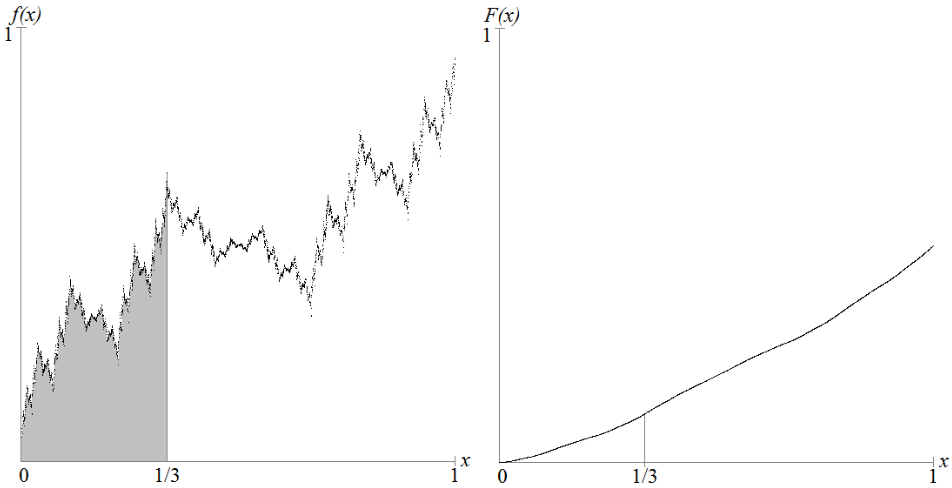

FIGURE 6. $F(x)$ corresponds to the area under the graph of f from 0 to x, or $\int_0^x f(t)\,dt$. (Graphs are approximate.)

We can see from the approximate graph that F appears to nondecreasing everywhere on $[0, 1]$. This is, in fact, the case, since $f(x) \geq 0$ for all $x \in [0, 1]$. We also can see that the graph looks perfectly smooth, but it also appears to shift between upwards and downwards concavity everywhere. The Fundamental Theorem of Calculus explains both of these observations. Okamoto [3] has proven that f is continuous and well-defined everywhere on $[0, 1]$, and according to the Fundamental Theorem of Calculus, that means that its antiderivative F is continuous, well-defined, and differentiable everywhere on $[0, 1]$; also, $F'(x) = f(x)$, so it follows that $F''(x) = f'(x)$. But Okamoto [3] has shown that $f'(x)$ does not exist for any $x \in [0, 1]$, so $F''(x)$ also does not exist for any $x \in [0, 1]$—in other words, the graph of F is neither concave up nor concave down anywhere on $[0, 1]$. This differs from the concavity of a line, as any linear function of the form $l(x) = ax + b$ will have a second derivative of $l''(x) = 0$ and thus could be said to be *both* concave up and concave down.

5. Integral Values

Like f, F has no explicit formula, so we must use our identities to predict different values of $F(x)$.

Theorem 5. *For $i > 0$,*

(i) $$\int_0^{1/(3^i+1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{3^i-1}{3^i+1}\,\frac{1}{1-(2/9)^i},$$

(ii) $$\int_0^{1/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{3^i+1}{3^i-1}\,\frac{1}{1+2^{i-1}/9^i},$$

(iii) $$\int_0^{2/(3^i+1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{5\cdot 3^i+1}{2\cdot 3^i+2}\,\frac{1}{1+2^{i-1}/9^i},\ \textit{and}$$

(iv) $$\int_0^{2/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{5\cdot 3^i-1}{2\cdot 3^i-2}\,\frac{1}{1-(2/9)^i}.$$

Proof. Let $i > 0$.

(i) Now, we know by Proposition 4 that for all $x \in [0,1]$ and $i \geq 0$,

$$\int_0^{x/3^i} f(t)\,dt = \left(\frac{2}{9}\right)^i \int_0^x f(t)\,dt$$

and clearly

$$1 - \frac{1}{3^i+1} = \frac{3^i}{3^i+1}$$

So keeping this in mind and applying Theorem 3, we have

$$\begin{aligned}\int_0^{1/(3^i+1)} f(t)\,dt &= \left(\frac{2}{9}\right)^i \int_0^{3^i/(3^i+1)} f(t)\,dt \\ &= \left(\frac{2}{9}\right)^i \left[\int_0^{1/(3^i+1)} f(t)\,dt + \int_{1/(3^i+1)}^{3^i/(3^i+1)} f(t)\,dt\right] \\ &= \left(\frac{2}{9}\right)^i \int_0^{1/(3^i+1)} f(t)\,dt + \left(\frac{2}{9}\right)^i \left(\frac{1}{2} - \frac{1}{3^i+1}\right)\end{aligned}$$

which means that

$$\left[1 - \left(\frac{2}{9}\right)^i\right] \int_0^{1/(3^i+1)} f(t)\,dt = \left(\frac{2^{i-1}}{9^i}\right)\left(\frac{3^i-1}{3^i+1}\right).$$

Since $1-(2/9)^i = 0$ when $i = 0$, we make the restriction $i > 0$ in order to divide on both sides. This gives us

$$\int_0^{1/(3^i+1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{3^i-1}{3^i+1}\,\frac{1}{1-(2/9)^i},\ \text{for } i > 0.$$

(ii) Now, we know by Proposition 5 that for all $x \in [0,1]$ and $i > 0$,

$$\int_{(2-x)/3^i}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[x + \int_0^x f(t)\,dt\right]$$

and clearly

$$1 - \frac{1}{3^i-1} = \frac{3^i-2}{3^i-1}.$$

Thus, we can see that

$$\int_{1/(3^i-1)}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[\frac{3^i-2}{3^i-1} + \int_0^{(3^i-2)/(3^i-1)} f(t)\,dt\right],$$

and so

$$\int_0^{2/3^i} f(t)\,dt - \int_0^{1/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[\frac{3^i-2}{3^i-1} + \int_0^{1/(3^i-1)} f(t)\,dt + \int_{1/(3^i-1)}^{(3^i-2)/(3^i-1)} f(t)\,dt\right].$$

Then

$$\int_0^{1/3^i} f(t)\,dt + \int_{1/3^i}^{2/3^i} f(t)\,dt - \int_0^{1/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\left(\frac{3^i-2}{3^i-1}\right) + \frac{2^{i-1}}{9^i}\int_0^{1/(3^i-1)} f(t)\,dt + \frac{2^{i-1}}{9^i}\left(\frac{1}{2}-\frac{1}{3^i-1}\right).$$

Propositions 4 and 5 give us

$$\left(\frac{1}{2}\right)\left(\frac{2}{9}\right)^i + \frac{2^{i-1}}{9^i}\left(1+\frac{1}{2}\right) - \int_0^{1/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\left(\frac{3^i-2}{3^i-1}\right) + \frac{2^{i-1}}{9^i}\int_0^{1/(3^i-1)} f(t)\,dt + \frac{2^{i-1}}{9^i}\left(\frac{1}{2}-\frac{1}{3^i-1}\right).$$

So

$$\begin{aligned}\left(1+\frac{2^{i-1}}{9^i}\right)\int_0^{1/(3^i-1)} f(t)\,dt &= \left(\frac{1}{2}\right)\left(\frac{2}{9}\right)^i + \frac{2^{i-1}}{9^i}\left(1+\frac{1}{2}\right) - \frac{2^{i-1}}{9^i}\left(\frac{3^i-2}{3^i-1}\right) \\ &\quad - \frac{2^{i-1}}{9^i}\left(\frac{1}{2}-\frac{1}{3^i-1}\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{2}{9}\right)^i + \frac{2^{i-1}}{9^i}\left(1-\frac{3^i-2}{3^i-1}+\frac{1}{3^i-1}\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{2}{9}\right)^i + \left(\frac{2}{9}\right)^i\left(\frac{1}{3^i-1}\right) \\ &= \frac{2^{i-1}}{9^i}\,\frac{3^i+1}{3^i-1}.\end{aligned}$$

Now,

$$\int_0^{1/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{3^i+1}{3^i-1}\,\frac{1}{1+2^{i-1}/9^i},\ \text{for } i>0.$$

(iii) We know by Proposition 5 that for all $x \in [0,1]$ and $i > 0$,

$$\int_{(2-x)/3^i}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[x + \int_0^x f(t)\,dt\right]$$

and we can see that clearly

$$\frac{2-2/(3^i+1)}{3^i} = \frac{2}{3^i+1}.$$

so

$$\begin{aligned}\int_{2/(3^i+1)}^{2/3^i} f(t)\,dt &= \int_0^{2/3^i} f(t)\,dt - \int_0^{2/(3^i+1)} f(t)\,dt \\ &= \int_0^{1/3^i} f(t)\,dt + \int_{1/3^i}^{2/3^i} f(t)\,dt - \int_0^{2/(3^i+1)} f(t)\,dt \\ &= \left(\frac{2}{9}\right)^i\left(\frac{1}{2}\right) + \frac{2^{i-1}}{9^i}\left(1+\frac{1}{2}\right) - \int_0^{2/(3^i+1)} f(t)\,dt\end{aligned}$$

and

$$\int_{2/(3^i+1)}^{2/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}\left[\frac{2}{3^i+1} + \int_0^{2/(3^i+1)} f(t)\,dt\right]$$
$$= \frac{2^{i-1}}{9^i}\left(\frac{2}{3^i+1}\right) + \frac{2^{i-1}}{9^i}\int_0^{2/(3^i+1)} f(t)\,dt.$$

Thus, by substitution,

$$\left(\frac{2}{9}\right)^i\left(\frac{1}{2}\right) + \frac{2^{i-1}}{9^i}\left(1+\frac{1}{2}\right) - \int_0^{2/(3^i+1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\left(\frac{2}{3^i+1}\right) + \frac{2^{i-1}}{9^i}\int_0^{2/(3^i+1)} f(t)\,dt,$$

and so

$$\left(1+\frac{2^{i-1}}{9^i}\right)\int_0^{2/(3^i+1)} f(t)\,dt = \left(\frac{2}{9}\right)^i\left(\frac{1}{2}\right) + \frac{2^{i-1}}{9^i}\left(\frac{3}{2}\right) - \frac{2^{i-1}}{9^i}\left(\frac{2}{3^i+1}\right)$$
$$= \frac{2^{i-1}}{9^i}\left(1+\frac{3}{2}-\frac{2}{3^i+1}\right)$$
$$= \frac{2^{i-1}}{9^i}\,\frac{5\cdot 3^i+1}{2\cdot 3^i+2}.$$

Therefore,

$$\int_0^{2/(3^i+1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{5\cdot 3^i+1}{2\cdot 3^i+2}\,\frac{1}{1+2^{i-1}/9^i},\ \text{for } i>0.$$

(iv) We know by Proposition 6 that for all $x \in [0,1]$ and $i > 0$,

$$\int_{2/3^i}^{(2+x)/3^i} f(t)\,dt = \frac{2^{i-1}}{9^i}x + \left(\frac{2}{9}\right)^i \int_0^x f(t)\,dt$$

and clearly

$$\frac{2+2/(3^i-1)}{3^i} = \frac{2}{3^i-1}.$$

The rest of the proof follows steps similar to those in part (iii) of this theorem to give us

$$\int_0^{2/(3^i-1)} f(t)\,dt = \frac{2^{i-1}}{9^i}\,\frac{5\cdot 3^i-1}{2\cdot 3^i-2}\,\frac{1}{1-(2/9)^i},\ \text{for } i>0. \qquad \square$$

6. Comments on Dimension

We have established that the graphs of f and F both exhibit self-similarity and pathological behavior. Nevertheless, if we are to consider either graph a fractal curve, we must show that its Hausdorff dimension strictly exceeds its topological dimension—which, in both cases, is one.

Because the graphs of f and F are the unique invariant sets of IFS consisting of three contraction mappings each, it might seem plausible to find their similarity dimension by solving for s in the equation

$$\sum_j r_j^s = 1,$$

where each r_j is a ratio of contraction. But we see that not all of the contraction mappings for the graphs can be written in the form $w(\mathbf{x}) = r\mathbf{x} + (1-r)\mathbf{a}$ (where $\mathbf{a}$ is the center of contraction), because not every one of these mappings has equal horizontal and vertical contraction ratios. So we must find the Hausdorff dimensions of the graphs of f and F another way.

In the two proofs that follow, we will use the *open set condition*: If $S_1(X), S_2(X), \ldots, S_m(X)$ are the components of a self-similar set X and if there exists a non-empty bounded set V such that

$$V \supset \bigcup_{j=1}^{m} S_j(V)$$

where the union is disjoint, then X satisfies the condition. In this case, the box-counting dimension $\dim_B(X)$ and Hausdorff dimension $\dim_H(X)$ of X are equal (see e.g., [5]).

Theorem 6. *If* Γ *is the graph of* f*, then its Hausdorff dimension* $\dim_H(\Gamma) = \log_3 5$.

Proof. Let Γ be the graph of f. We must look at f as $\lim_{i\to\infty} f_i$ here, so we will define Γ_i as the graph of f_i. Obviously, $\Gamma = \lim_{i\to\infty} \Gamma_i$.

Now we consider the box-counting dimension $\dim_B(\Gamma_i)$ of Γ_i. As the affine pieces of Γ_i are defined over intervals of length $1/3^i$, we will count how many boxes of side length $\delta = 1/3^i$ will cover Γ_i. For Γ_0, the graph of $y = x$ for $x \in [0,1]$, the number N_δ of boxes required for the cover is clearly 1; with $\delta = 1$, we have a box covering the entire graph. For Γ_1, we count boxes with $\delta = 1/3$, and we get $N_\delta = 5$; exactly two boxes cover each of the graph's "tall" sides, and one box covers the central portion. For Γ_2 with boxes of side length $\delta = 1/9$, we have $N_\delta = 25$; we count four boxes for each of the four tallest sections, two boxes for each of the four second-tallest sections, and one for the centermost section (see Fig. 7).

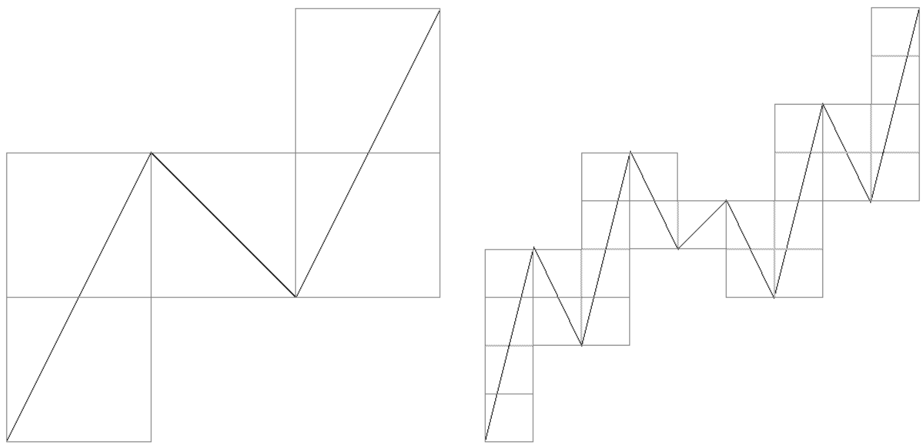

FIGURE 7. Box-counting for Γ_1 and Γ_2

This gives us an idea of how i varies with N_δ, but to obtain a general result, we will take the route of Katsuura [4] again and view Γ as the attractor for a three-component IFS. We recall that all three contraction mappings from Γ_i to Γ_{i+1} shinks Γ_i horizontally by a factor of $1/3$, but the first and third shrink Γ_i vertically by a factor of $2/3$, while the second does so by a factor of only $1/3$. Now, if we can cover Γ_i by $N_\delta(\Gamma_i)$ boxes of side length $\delta = 1/3^i$, then by necessity, the middle portion of Γ_{i+1}—the region of the second mapping—could be covered by $N_\delta(\Gamma_i)$ boxes, as well, since it is Γ_i scaled down by a factor of $1/3$ and we are counting how many boxes scaled down by the same factor can cover it. Applying the similar logic to the regions of the first and third mappings, we can see that Γ_i scaled down horizontally by 1/3 and vertically by 2/3 will be covered by $2N_\delta(\Gamma_i)$ boxes scaled down by a factor of 1/3. Therefore,

$$\begin{aligned} N_\delta(\Gamma_{i+1}) &= 2N_\delta(\Gamma_i) + N_\delta(\Gamma_i) + 2N_\delta(\Gamma_i) \\ &= 5N_\delta(\Gamma_i) \end{aligned}$$

and since $N_\delta(\Gamma_0)$ is 1, we can say that for $i > 0$,

$$N_\delta(\Gamma_i) = 5^i.$$

Now we consider the formula for box-counting dimension:

$$\dim_B(\Gamma) = \lim_{\delta \to 0} \frac{\log(N_\delta)}{-\log(\delta)} \text{ (See e.g., [5]).}$$

And since $\delta = 1/3^i$, the formula becomes

$$\begin{aligned} \dim_B(\Gamma) &= \lim_{i \to \infty} \frac{\log(N_\delta(\Gamma_i))}{-\log(1/3^i)} \\ &= \lim_{i \to \infty} \frac{\log(5^i)}{\log(3^i)} \\ &= \log_3 5. \end{aligned}$$

So the box-counting dimension of Γ is $\log_3 5$. To show that this is equivalent to the Hausdorff dimension, we will show that Γ satisfies the open-set condition. Again, we look at Γ as the unique invariant set of an IFS with three components. Each of the three contractions maps the image of Γ_i to a separate third of Γ_{i+1}, so if we treat the regions to which the contractions map Γ_i as open sets, we can see that the three regions of contraction do not overlap at all. Their union is disjoint and covers every part of Γ. This means that Γ satisfies the open-set condition, and therefore, $\dim_H(\Gamma) = \dim_B(\Gamma) = \log_3 5$. □

So Γ is, by definition, a fractal. But the same is not true of the graph of the antiderivative F, as we will show.

Theorem 7. *If G is the graph of F, then its Hausdorff dimension* $\dim_H(G) = 1$.

Proof. Let G be the graph of F. First, we will show that G has a finite arc length. We must consider F as the limit of its iterations here, so we will define G_i as the graph of F_i. Obviously, $G = \lim_{i\to\infty} G_i$.

Because each F_i is affine on $[0, 1/3^i], [1/3^i, 2/3^i], \ldots, [(3^i - 1)/3^i, 1]$ we can apply the Triangle Inequality to the linear "piece" of G_i at each of these intervals; for instance, if l is the length of G_i on $[1/3^i, 2/3^i]$, we have $|2/3^i - 1/3^i| + |F(2/3^i) - F(1/3^i)| \geq l$ (see Fig. 8).

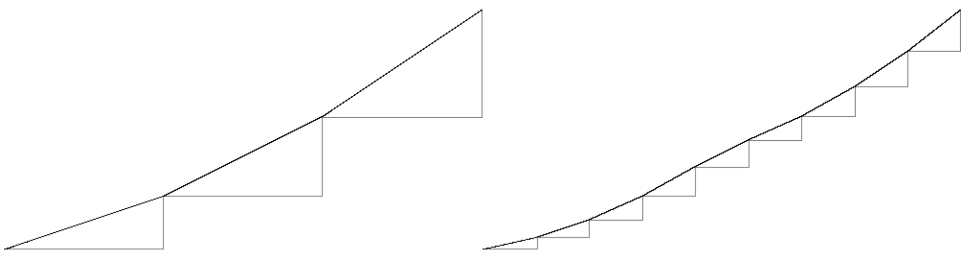

FIGURE 8. Applying the Triangle Inequality to G_1 and G_2

Because the inequality holds over all of $[0,1/3^i],[1/3^i,2/3^i],\ldots,[(3^i-1)/3^i,1]$, it also will hold for the sums of the respective sides of each "triangle"; that is, if L_i is the total arc length of G_i, then

$$L_i \leq \sum_{j=1}^{3^i}\left|\frac{j}{3^i}-\frac{j-1}{3^i}\right| + \sum_{j=1}^{3^i}\left|F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)\right|.$$

So by taking the limit as i approaches infinity on both sides, we get

$$\begin{aligned} L &\leq \lim_{i\to\infty}\sum_{j=1}^{3^i}\left|\frac{j}{3^i}-\frac{j-1}{3^i}\right| + \sum_{j=1}^{3^i}\left|F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)\right| \\ &\leq \lim_{i\to\infty}\sum_{j=1}^{3^i}\frac{1}{3^i} + \sum_{j=1}^{3^i}\left|F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)\right| \\ &\leq \lim_{i\to\infty} 1 + \sum_{j=1}^{3^i}\left|F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)\right|. \end{aligned}$$

We observed earlier that F is nondecreasing. Now, $F_{i+1}(k/3^i) = F_i(k/3^i)$, so by induction, $F(x) = F_i(x)$ wherever $x = k/3^i$. And since every F_i is affine everywhere between such points, every F_i must be nondecreasing everywhere on $[0,1]$, as well. This means that for all $j \in \{1,2,\ldots,3^i\}$,

$$F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right) \geq 0$$

and thus, we can make the substitution

$$\left|F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)\right| = F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)$$

which gives us

$$\begin{aligned} L &\leq \lim_{i\to\infty} 1 + \sum_{j=1}^{3^i} F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right) \\ &\leq \lim_{i\to\infty} 1 + F_i\left(\frac{1}{3^i}\right)-F_i\left(\frac{0}{3^i}\right)+F_i\left(\frac{2}{3^i}\right)-F_i\left(\frac{1}{3^i}\right) \\ &\quad + \cdots + F_i\left(\frac{3^i}{3^i}\right)-F_i\left(\frac{3^i-1}{3^i}\right) \\ &\leq \lim_{i\to\infty} 1 + F_i(1) - F_i(0) \\ &\leq \lim_{i\to\infty} 1 + \frac{1}{2} - 0 \\ &\leq \frac{3}{2}. \end{aligned}$$

Applying the same process to the inequality

$$\sum_{j=1}^{3^i}\left|\frac{j}{3^i}-\frac{j-1}{3^i}\right| \leq L_i + \sum_{j=1}^{3^i}\left|F_i\left(\frac{j}{3^i}\right)-F_i\left(\frac{j-1}{3^i}\right)\right|$$

gives us

$$L \geq \frac{1}{2}$$

and so we have

$$\frac{1}{2} \leq L \leq \frac{3}{2}.$$

To show that G has Hausdorff dimension 1, we first will show that it has box-counting dimension 1. We will consider a variant of the standard box-counting dimension in which we count the number N_δ of disjoint balls of diameter δ needed to cover G. The dimension in this case can be expressed in the same way as the conventional box-counting dimension:

$$\dim_B(G) = \lim_{\delta \to 0} \frac{\log(N_\delta)}{-\log(\delta)}$$

Given that our arc length L is finite, we know that $N_\delta = L/\delta$. So we have

$$\begin{aligned}\dim_B(G) &= \lim_{\delta \to 0} \frac{\log(L/\delta)}{-\log(\delta)} \\ &= \lim_{\delta \to 0} \frac{\log(L) - \log(\delta)}{-\log(\delta)} \\ &= \lim_{\delta \to 0} 1 + \frac{\log(L)}{-\log(\delta)}\end{aligned}$$

After we substitute our upper and lower bounds for L, our formula becomes

$$\lim_{\delta \to 0} 1 + \frac{\log(1/2)}{-\log(\delta)} \leq \dim_B(G) \leq \lim_{\delta \to 0} 1 + \frac{\log(3/2)}{-\log(\delta)}$$

And taking the limit gives us

$$1 \leq \dim_B(G) \leq 1$$

Therefore, the box-counting dimension of G is 1. Now, if we treat (9), (10), and (11) as an IFS whose unique invariant set is G, we see that these functions resemble (5), (6), and (7) for f inasmuch as they operate as contraction mappings for each G_i. Specifically, they map each G_i to three disjoint portions of $[0, 1] \times [0, 1/2]$: (9) maps to $[0, 1/3] \times [0, 1/2]$, (10) maps to $[1/3, 2/3] \times [0, 1/2]$, and (11) maps to $[2/3, 1] \times [0, 1/2]$. So G satisfies the open set condition, and its box-counting dimension and its Hausdorff dimension are equal. Therefore, G has Hausdorff dimension 1. □

We conclude that although the graph of F exhibits self-similarity and pathological behavior, it is by definition not a fractal. If we see F as a measure of the area bounded by the graph of f and the x-axis, this conclusion makes more sense; a region with a fractal boundary of infinite length still can contain a finite area. (A standard example of this phenomenon is the Koch snowflake; see e.g., [6].)

7. Concluding Remarks

Okamoto [3] shows that Bourbaki's Function is just one member of a family of functions with analogous constructions. For this reason, it would not be unreasonable to think that the other functions in this family will have similar identities as the ones we have decsribed for Bourbaki's Function. Not all functions in this family are nowhere differentiable, however—a fact that may influence several properties of these functions, including the dimension of their graphs and the nature of their antiderivatives. We expect more general proofs to shed light on these subjects.

Obviously, the formulas for finding function and integral values in Theorems 2 and 4 do not guarantee results for any number in $[0, 1]$ or even any rational number in that interval. We don't know of any shortcut for finding $f(1/7)$, for instance, since $1/7$ cannot be expressed in terms of $1/(3^i+1)$, $1/(3^i-1)$, $2/(3^i+1)$, $2/(3^i-1)$, $1/(3^j + 3^i)$, or $1/(3^j - 3^i)$. We are unsure if a simple algorithm can be found for evaluating $f(1/m)$ for any natural number m; while we attempted to do this by parts for $f(1/3m)$, $f(1/[3m - 1])$, and $f(1/[3m - 2])$, we were unsuccessful.

8. Acknowledgments

The approximate graphs of f and F were produced using *Dynamical Grapher for Quadratic Maps* [7].

Many thanks to Professor Dan Jackson for providing resources, feedback, and encouragment throughout the duration of this project.

References

[1] Jarník, Vojtêch, and Bernard Bolzano. *Bolzano and the Foundations of Mathematical Analysis.* Prague: Society of Czechoslovak Mathematicians and Physicists, 1981.

[2] Bourbaki, Nicolas. *Functions of a Real Variable: Elementary Theory.* Trans. Philip Spain. Berlin: Springer, 2004.

[3] Okamoto, Hisashi. "A remark on continuous, nowhere differentiable functions." *Proceedings of the Japan Academy* A 81.3 (2005) 47–50.

[4] Katsuura, Hidefumi. "Continuous Nowhere-Differentiable Functions—an Application of Contraction Mappings." *American Mathematical Monthly* 98.5 (1991) 411–416.

[5] Falconer, Kenneth. *Fractal Geometry—Mathematical Foundations and Applications.* Chichester: Wiley, 2003.

[6] Edgar, Gerald. *Measure, Topology, and Fractal Geometry.* New York, NY: Springer, 2008.

[7] Basselet, Hunter, Adam Case, and Daniel Jackson. *Dynamical Grapher for Quadratic Maps.* Computer software. Vers. 1.0. Web. 25 July 2010. <http://faculty.umf.maine.edu/daniel.jackson1/public.www/DGrapher/ DGrapher.html>.

Division of Mathematics and Computer Science, University of Maine at Farmington, Farmington, ME 04938

E-mail address: `james.mccollum@maine.edu`